Czech-Polish-Slovak Match 2018

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by Amir Hossein

1 Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for all real numbers x and y,

$$f(x^2+xy)=f(x)f(y)+yf(x)+xf(x+y).$$

Proposed by Walther Janous, Austria

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2 Let ABC be an acute scalene triangle. Let D and E be points on the sides AB and AC, respectively, such that $BD=CE$. Denote by O_1 and O_2 the circumcentres of the triangles ABE and ACD, respectively. Prove that the circumcircles of the triangles ABC, ADE, and AO_1O_2 have a common point different from A.

Proposed by Patrik Bak, Slovakia

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3 There are 2018 players sitting around a round table. At the beginning of the game we arbitrarily deal all the cards from a deck of K cards to the players (some players may receive no cards). In each turn we choose a player who draws one card from each of the two neighbors. It is only allowed to choose a player whose each neighbor holds a nonzero number of cards. The game terminates when there is no such player. Determine the largest possible value of K such that, no matter how we deal the cards and how we choose the players, the game always terminates after a finite number of turns.

Proposed by Peter Novotn, Slovakia

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4 Let ABC be an acute triangle with the perimeter of $2s$. We are given three pairwise disjoint circles with pairwise disjoint interiors with the centers A, B, and C, respectively. Prove that there exists a circle with the radius of s which contains all the three circles.

Proposed by Josef Tkadlec, Czechia

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5 In a 2×3 rectangle there is a polyline of length 36, which can have self-intersections. Show that there exists a line parallel to two sides of the rectangle, which intersects the other two sides in their interior points and intersects the polyline in fewer than 10 points.

Proposed by Josef Tkadlec, Czechia and Vojtech Blint, Slovakia

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6 We say that a positive integer n is *fantastic* if there exist positive rational numbers a and b such that

$$n=a+\frac{1}{a}+b+\frac{1}{b}.$$

(a) Prove that there exist infinitely many prime numbers p such that no multiple of p is fantastic.
(b) Prove that there exist infinitely many prime numbers p such that some multiple of p is fantastic.

Proposed by Walther Janous, Austria

Source Taken from here (https://skmo.sk/dokument.php?id=3017). Including official solutions in English.
