Pan-African Mathematical Olympiad problems from 2018

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1 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$(f(x+y))^2 = f(x^2) + f(y^2)$$

for all $x, y \in \mathbb{Z}$.

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2 A chess tournament is held with the participation of boys and girls. The girls are twice as many as boys. Each player plays against each other player exactly once. By the end of the tournament, there were no draws and the ratio of girl winnings to boy winnings was $\frac{7}{9}$.

How many players took part at the tournament?

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3 For any positive integer x, we set

$$g(x) = \text{ largest odd divisor of } x,$$

$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even;} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Consider the sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the integer 2018 appears in this sequence, determine the least integer n such that $x_n = 2018$, and determine whether n is unique or not.

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4 Given a triangle ABC, let D be the intersection of the line through A perpendicular to AB, and the line through B perpendicular to BC. Let P be a point inside the triangle. Show that $DAPB$ is cyclic if and only if $\angle BAP = \angle CBP$.

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5 Let a, b, c and d be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ and } ac = bd.$$

Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leq -12$$

and that -12 is the maximum.

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6 A circle is divided into n sectors ($n \geq 3$). Each sector can be filled in with either 1 or 0. Choose any sector $\mathcal{C}$ occupied by 0, change it into a 1 and simultaneously change the symbols x, y in the two sectors adjacent to $\mathcal{C}$ to their complements $1-x$, $1-y$. We repeat this process as long as there exists a zero in some sector. In the initial configuration there is a 0 in one sector and 1s elsewhere. For which values of n can we end this process?