National Science Olympiad 2018

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–	Day 1
1	Let a be a positive integer such that $\gcd(an+1, 2n+1) = 1$ for all integer n. a) Prove that $\gcd(a-2, 2n+1) = 1$ for all integer n. b) Find all possible a.
2	Let Γ_1, Γ_2 be circles that touch at a point A, and Γ_2 is inside Γ_1. Let B be on Γ_2, and let AB intersect Γ_1 on C. Let D be on Γ_1 and P be on the line CD (may be outside of the segment CD). BP intersects Γ_2 at Q. Prove that A, D, P, Q lie on a circle.
3	Alzim and Badril are playing a game on a hexagonal lattice grid with 37 points (4 points a side), all of them uncolored. On his turn, Alzim colors one uncolored point with the color red, and Badril colors **two** uncolored points with the color blue. The game ends either when there is an equilateral triangle whose vertices are all red, or all points are colored. If the former happens, then Alzim wins, otherwise Badril wins. If Alzim starts the game, does Alzim have a strategy to guarantee victory?
4	In a game, Andi and a computer take turns. At the beginning, the computer shows a polynomial $x^2 + mx + n$ where $m, n \in \mathbb{Z}$, such that it doesn't have real roots. Andi then begins the game. On his turn, Andi may change a polynomial in the form $x^2 + ax + b$ into either $x^2 + (a+b)x + b$ or $x^2 + ax + (a+b)$. However, Andi may only choose a polynomial that has real roots. On the computer's turn, it simply switches the coefficient of x and the constant of the polynomial. Andi loses if he can't continue to play. Find all (m, n) such that Andi always loses (in finitely many turns).
–	Day 2
5	Find all triples of reals (x, y, z) satisfying: $$\begin{cases} \frac{1}{3}\min\{x,y\} + \frac{2}{3}\max\{x,y\} = 2017 \\ \frac{1}{3}\min\{y,z\} + \frac{2}{3}\max\{y,z\} = 2018 \\ \frac{1}{3}\min\{z,x\} + \frac{2}{3}\max\{z,x\} = 2019 \end{cases}$$
6	Find all prime numbers p such that there exists a positive integer n where $2^n p^2 + 1$ is a square number.

7 Suppose there are three empty buckets and $n \geq 3$ marbles. Ani and Budi play a game. For the first turn, Ani distributes all the marbles into the buckets so that each bucket has at least one marble. Then Budi and Ani alternate turns, starting with Budi. On a turn, the current player may choose a bucket and take 1, 2, or 3 marbles from it. The player that takes the last marble wins. Find all n such that Ani has a winning strategy, including what Ani's first move (distributing the marbles) should be for these n.

8 Let I, O be the incenter and circumcenter of the triangle ABC respectively. Let the excircle ω_A of ABC be tangent to the side BC on N, and tangent to the extensions of the sides AB, AC on K, M respectively. If the midpoint of KM lies on the circumcircle of ABC, prove that O, I, N are collinear.