Problems from the 2008 iTest Tournament of Champions

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by djmathman

– Round 1

1 Find k where 2^k is the largest power of 2 that divides the product

$$2008 \cdot 2009 \cdot 2010 \cdots 4014.$$

2 Let

$$\begin{array}{rcrcrcrcccc} A & = & 5\cdot 6 & - & 6\cdot 7 & + & 7\cdot 8 & - & \cdots & + & 2003\cdot 2004, \\ B & = & 1\cdot 10 & - & 2\cdot 11 & + & 3\cdot 12 & - & \cdots & + & 1999\cdot 2008. \end{array}$$

Find the value of $A - B$.

3 Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the rst game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the nal hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n. Find the sum of all possible values of n.

4 The rules for the movement of a king on a chessboard are as follows: The king can legally move to any of the (up to 8) squares adjacent diagonally or on a side. Andrew places a king on an ordinary 8×8 chessboard. He then makes 64 total moves with the king such that the king visits every square on the board, never crosses its own path, and winds up at its original position (where Andrew rst placed it). Along the way, Andrew counts the number of times the king moves diagonally (from one square to another that shares no side). Call that number M. Find the maximum possible value of M.

5 Let $c_1, c_2, c_3, \ldots, c_{2008}$ be complex numbers such that

$$|c_1| = |c_2| = |c_3| = \cdots = |c_{2008}| = 1492,$$

and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of

$$\left|\frac{S(2008, 1492)}{S(2008, 516)}\right|.$$

Find the remainder when Q is divided by 2008.

– Round 2

1 Find the smallest positive integer n such that there are at least three distinct ordered pairs (x, y) of positive integers such that
$$x^2 - y^2 = n.$$

2 Find the value of $|xy|$ given that x and y are integers and
$$6x^2y^2 + 5x^2 - 18y^2 = 17253.$$

3 A regular 2008-gon is located in the Cartesian plane such that $(x_1, y_1) = (p, 0)$ and $(x_{1005}, y_{1005}) = (p + 2, 0)$, where p is prime and the vertices,
$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_{2008}, y_{2008}),$$
are arranged in counterclockwise order. Let
$$S = (x_1 + y_1 i)(x_3 + y_3 i)(x_5 + y_5 i) \cdots (x_{2007} + y_{2007} i),$$
$$T = (y_2 + x_2 i)(y_4 + x_4 i)(y_6 + x_6 i) \cdots (y_{2008} + x_{2008} i).$$
Find the minimum possible value of $|S - T|$.

4 Find the maximum of $x + y$ given that x and y are positive real numbers that satisfy
$$x^3 + y^3 + (x + y)^3 + 36xy = 3456.$$

5 While running from an unrealistically rendered zombie, Willy Smithers runs into a vacant lot in the shape of a square, 100 meters on a side. Call the four corners of the lot corners 1, 2, 3, and 4, in clockwise order. For $k = 1, 2, 3, 4$, let d_k be the distance between Willy and corner k. Let

(a) $d_1 < d_2 < d_4 < d_3$,

(b) d_2 is the arithmetic mean of d_1 and d_3, and

(c) d_4 is the geometric mean of d_2 and d_3.

If d_1^2 can be written in the form $\dfrac{a - b\sqrt{c}}{d}$, where $a, b, c,$ and d are positive integers, c is square-free, and the greatest common divisor of a, b, and d is 1, find the remainder when $a + b + c + d$ is divided by 2008.

– Round 3

1 Find the remainder when $712!$ is divided by 719.

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2 Note that there are exactly three ways to write the integer 4 as a sum of positive odd integers where the order of the summands matters:

$$\begin{aligned}1+1+1+1&=4,\\ 1+3&=4,\\ 3+1&=4.\end{aligned}$$

Let $f(n)$ be the number of ways to write a natural number n as a sum of positive odd integers where the order of the summands matters. Find the remainder when $f(2008)$ is divided by 100.

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3 Arthur stands on a circle drawn with chalk in a parking lot. It is sunrise and there are birds in the trees nearby. He stands on one of five triangular nodes that are spaced equally around the circle, wondering if and when the aliens will pick him up and carry him from the node he is standing on. He flips a fair coin 12 times, each time chanting the name of a nearby star system. Each time he flips a head, he walks around the circle, in the direction he is facing, until he reaches the next node in that direction. Each time he flips a tail, he reverses direction, then walks around the circle until he reaches the next node in that new direction. After 12 flips, Arthur finds himself on the node at which he started. He thinks this is fate, but Arthur is quite mistaken. If a and b are relatively prime positive integers such that a/b is the probability that Arthur flipped exactly 6 heads, find $a+b$.

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4 Euclid places a morsel of food at the point $(0,0)$ and an ant at the point $(1,2)$. Every second, the ant walks one unit in one of the four coordinate directions. However, whenever the ant moves to $(x,\pm 3)$, Euclid's malicious brother Mobius picks it up and puts it at $(-x,\mp 2)$, and whenever it moves to $(\pm 2,y)$, his cousin Klein puts it at $(\mp 1,y)$. If p and q are relatively prime positive integers such that $\frac{p}{q}$ is the expected number of steps the ant takes before reaching the food, find $p+q$.

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5 It is well-known that the n^{th} triangular number can be given by the formula $n(n+1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a,b,c) such that $a^2+b^2=c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a,b,c) such that $a\leq b<c$ and

$$\dfrac{a(a+1)}{2}+\dfrac{b(b+1)}{2}=\dfrac{c(c+1)}{2}.$$

For instance, $(3,5,6)$ is a PTTN ($6+15=21$). Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

– Round 4

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1 Yatta and Yogi play a game in which they begin with a pile of n stones. The players take turns removing 1, 2, 3, 5, 6, 7, or 8 stones from the pile. That is, when it is a player's turn to remove stones, that player may remove from 1 to 8 stones, but *cannot* remove exactly 4 stones. The player who removes the last stone *loses*. Yogi goes first and finds that he has a winning position, meaning that so long as he plays perfectly, Yatta cannot defeat him. For how many positive integers n from 100 to 2008 inclusive is this the case?

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2 Let A be the number of 12-digit words that can be formed by from the alphabet $\{0,1,2,3,4,5,6\}$ if each pair of neighboring digits must differ by exactly 1. Find the remainder when A is divided by 2008.

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3 For how many integers $1\leq n\leq 9999$ is there a solution to the congruence

$$\phi(n)\equiv 2\pmod{12},$$

where $\phi(n)$ is the Euler phi-function?

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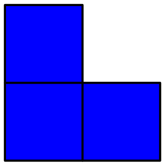
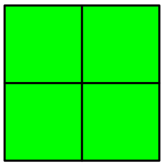
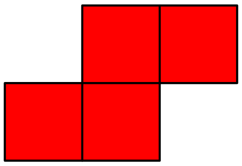
4 Each of the 24 students in Mr. Friedman's class cut up a 7×7 grid of squares while he read them short stories by Mark Twain. While not all of the students cut their squares up in the same way, each of them cut their 7×7 square into at most the three following types (shapes) of pieces.







Let a, b, and c be the number of total pieces of each type from left to right respectively after all 24 7×7 squares are cut up. How many ordered triples (a,b,c) are possible?

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5 For positive integers $m,n\geq 3$, let $h(m,n)$ be the maximum (finite) number of intersection points between a simple m-gon and a simple n-gon. (Note: a polygon is simple if it does not intersect itself.) Evaluate

$$\sum_{m=3}^6\sum_{n=3}^{12}h(m,n).$$

– Round 5

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1 Let a, b, c, and d be positive real numbers such that $abcd = 17$. Let m be the minimum possible value of
$$a^2 + b^2 + c^2 + a(b + c + d) + b(c + d) + cd.$$
Compute $\lfloor 17m \rfloor$.

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2 Let N be the smallest natural number that, when written to its left, forms an integer with twice as many digits that is a perfect square. Find the remainder when N is divided by 1000.

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3 The 260 volumes of the *Encyclopedia Galactica* are out of order in the library. Fortunately for the librarian, the books are numbered. Due to his religion, which holds both encyclopedias and the concept of parity in high esteem, the librarian can only rearrange the books two at a time, and then only by switching the position of an even numbered volume with that of an odd numbered volume. Find the minimum number of such transpositions sufficient to get the books back into ordinary sequential order, regardless of the starting positions of the books. (Find the minimum number of transpositions in the worst-case scenario.)

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4 Let
$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{1}{n-k} \binom{n-k}{k},$$
for each positive integer n. If $|f(2007) + f(2008)| = a/b$ for relatively prime positive integers a and b, find the remainder when a is divded by 1000.

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5 Two squares of side length 2 are glued together along their boundary so that the four vertices of the first square are glued to the midpoints of the four sides of the other square, and vice versa. This gluing results in a convex polyhedron. If the square of the volume of this polyhedron is written in simplest form as $\frac{a+b\sqrt{c}}{d}$, what is the value of $a + b + c + d$?

– Round 6

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1 Let
$$X = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cdots + \cos\frac{2006\pi}{7} + \cos\frac{2008\pi}{7}.$$
Compute $\Big|\lfloor 2008X \rfloor\Big|$.

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2 Jon wrote the n smallest perfect squares on one sheet of paper, and the n smallest triangular numbers on another (note that 0 is both square and triangular). Jon notices that there are the same number of triangular numbers on the first paper as there are squares on the second paper, but if n had been one smaller, this would not have been true. If $n < 2008$, let m be the greatest number Jon could have written on either paper. Find the remainder when m is divided by 2008.

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3 Let $\phi = \frac{1+\sqrt{5}}{2}$ be the positive root of $x^2 = x + 1$. Define a function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(0) = 1$$
$$f(2x) = \lfloor \phi f(x) \rfloor$$
$$f(2x+1) = f(2x) + f(x).$$

Find the remainder when $f(2007)$ is divided by 2008.

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4 If m is a positive integer, let S_m be the set of rational numbers in reduced form with denominator at most m. Let $f(m)$ be the sum of the numerator and denominator of the element of S_m closest to e (Euler's constant). Given that $f(2007) = 3722$, find the remainder when $f(1000)$ is divided by 2008.

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5 Three circles with centers V_0, V_1, V_2 and radii $33, 30, 25$ respectively and mutually externally tangent: P_i is the tangency point between circles V_{i+1} and V_{i+2}, where indeces are taken modulo 3. For $i = 0, 1, 2$, line $P_{i+1}P_{i+2}$ intersects circle V_{i+1} at P_{i+2} and Q_i, and the same line intersects circle V_{i+2} at P_{i+1} and R_i. If Q_0R_1 intersects Q_2R_0 at X, then the distance from X to line R_1Q_2 can be expressed as $\frac{a\sqrt{b}}{c}$, where the integer b is not divisible by the square of any prime, and positive integers a and c are relatively prime. Find the value of $b + c$.
