Saint Petersburg Mathematical Olympiad 2018

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–	**Grade 11**
1	Let l some line, that is not parallel to the coordinate axes. Find minimal d that always exists point A with integer coordinates, and distance from A to l is $\leq d$
2	Vasya has 100 cards of 3 colors, and there are not more than 50 cards of same color. Prove that he can create 10×10 square, such that every cards of same color have not common side.
3	Point T lies on the bisector of $\angle B$ of acuteangled $\triangle ABC$. Circle S with diameter BT intersects AB and BC at points P and Q. Circle, that goes through point A and tangent to S at P intersects line AC at X. Circle, that goes through point C and tangent to S at Q intersects line AC at Y. Prove, that $TX=TY$
4	$$(b+c)x^2+(a+c)x+(a+b)=0$$ has not real roots. Prove that $$4ac-b^2 \leq 3a(a+b+c)$$
5	Regular hexagon is divided to equal rhombuses, with sides, parallels to hexagon sides. On the three sides of the hexagon, among which there are no neighbors, is set directions in order of traversing the hexagon against hour hand. Then, on each side of the rhombus, an arrow directed just as the side of the hexagon parallel to this side. Prove that there is not a closed path going along the arrows.
6	α,β are positive irrational numbers and $[\alpha[\beta x]]=[\beta[\alpha x]]$ for every positive x. Prove that $\alpha=\beta$
7	Points A,B lies on the circle S. Tangent lines to S at A and B intersects at C. M -midpoint of AB. Circle S_1 goes through M,C and intersects AB at D and S at K and L. Prove, that tangent lines to S at K and L intersects at point on the segment CD.
–	**Grade 10**
1	Misha came to country with n cities, and every 2 cities are connected by the road. Misha want visit some cities, but he doesn't visit one city two time. Every time, when Misha goes from

city A to city B, president of country destroy k roads from city B(president can't destroy road, where Misha goes). What maximal number of cities Misha can visit, no matter how president does?

2 Color every vertex of 2008-gon with two colors, such that adjacent vertices have different color. If sum of angles of vertices of first color is same as sum of angles of vertices of second color, than we call 2008-gon as interesting.
Convex 2009-gon one vertex is marked. It is known, that if remove any unmarked vertex, then we get interesting 2008-gon. Prove, that if we remove marked vertex, then we get interesting 2008-gon too.

3 n coins lies in the circle. If two neighbour coins lies both head up or both tail up, then we can flip both. How many variants of coins are available that can not be obtained from each other by applying such operations?

4 $f(x)$ is polynomial with integer coefficients, with module not exceeded $5*10^6$. $f(x)=nx$ has integer root for $n=1,2,...,20$. Prove that $f(0)=0$

5 $ABCD$ is inscribed quadrilateral. Line, that perpendicular to BD intersects segments AB and BC and rays DA, DC at P,Q,R,S . $PR=QS$. M is midpoint of PQ. Prove that $AM=CM$

6 Let $a,b,c,d>0$. Prove that $a^4+b^4+c^4+d^4 \geq 4abcd+4(a-b)^2\sqrt{abcd}$

7 The checker moves from the lower left corner of the board 100×100 to the right top corner, moving at each step one cell to the right or one cell up. Let a be the number of paths in which exactly 70 steps the checker take under the diagonal going from the lower left corner to the upper right corner, and b is the number of paths in which such steps are exactly 110. What is more: a or b?

– **Grade 9**

1 Prove, that for every natural N exists k, such that $N=a_02^0+a_12^1+...+a_k2^k$, where $a_0,a_1,...a_k$ are 1 or 2

2 $n>1$ is odd number. There are numbers $n,n+1,n+2,...,2n-1$ on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

3 ABC is acuteangled triangle. Variable point X lies on segment AC, and variable point Y lies on the ray BC but not segment BC, such that $\angle ABX+\angle CXY=90$. T is projection of B on the XY. Prove that all points T lies on the line.

4 On the round necklace there are $n>3$ beads, each painted in red or blue. If a bead has adjacent beads painted the same color, it can be repainted (from red to blue or from blue to red). For what n for any initial coloring of beads it is possible to make a necklace in which all beads are painted equally?

5 Can we draw $\triangle ABC$ and points X, Y, such that $AX = BY = AB$, $BX = CY = BC$, $CX = AY = CA$?

6 a, b are odd numbers. Prove, that exists natural k that $b^k - a^2$ or $a^k - b^2$ is divided by 2^{2018}.

7 In 10×10 square we choose n cells. In every chosen cell we draw one arrow from the angle to opposite angle. It is known, that for any two arrows, or the end of one of them coincides with the beginning of the other, or
the distance between their ends is at least 2. What is the maximum possible value of n?