India International Mathematical Olympiad Training Camp 2018

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–	Practice Tests
–	Practice Test 1
1	Let ΔABC be an acute triangle. D, E, F are the touch points of incircle with BC, CA, AB respectively. AD, BE, CF intersect incircle at K, L, M respectively. If, $$\sigma = \frac{AK}{KD} + \frac{BL}{LE} + \frac{CM}{MF}$$ $$\tau = \frac{AK}{KD} . \frac{BL}{LE} . \frac{CM}{MF}$$ Then prove that $\tau = \frac{R}{16r}$. Also prove that there exists integers u, v, w such that, $uvw \neq 0$, $u\sigma + v\tau + w = 0$.
2	A 10 digit number is called interesting if its digits are distinct and is divisible by 11111. Then find the number of interesting numbers.
3	Let a_n, b_n be sequences of positive reals such that, $$a_{n+1} = a_n + \frac{1}{2b_n}$$ $$b_{n+1} = b_n + \frac{1}{2a_n}$$ for all $n \in \mathbb{N}$. Prove that, max $(a_{2018}, b_{2018}) > 44$.
–	Practice Test 2
1	Let $ABCD$ be a convex quadrilateral inscribed in a circle with center O which does not lie on either diagonal. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.
2	For an integer $n \geq 2$ find all $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ so that (a) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1 \leq b_1 \leq b_2 \leq \cdots \leq b_n$; (b) $\sum_{k=1}^{n}(a_k + b_k) = 2n$; (c) $\sum_{k=1}^{n}(a_k^2 + b_k^2) = n^2 + 3n$.

3 A convex polygon has the property that its vertices are coloured by three colors, each colour occurring at least once and any two adjacent vertices having different colours. Prove that the polygon can be divided into triangles by diagonals, no two of which intersect in the **interior** of the polygon, in such a way that all the resulting triangles have vertices of all three colours.

– Team Selection Tests

– TST 1

1 A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Proposed by Jeck Lim, Singapore

2 Let A, B, C be three points in that order on a line ℓ in the plane, and suppose $AB > BC$. Draw semicircles Γ_1 and Γ_2 respectively with AB and BC as diameters, both on the same side of ℓ. Let the common tangent to Γ_1 and Γ_2 touch them respectively at P and Q, $P \neq Q$. Let D and E be points on the segment PQ such that the semicircle Γ_3 with DE as diameter touches Γ_2 in S and Γ_1 in T.
-Prove that A, C, S, T are concyclic.
-Prove that A, C, D, E are concyclic.

3 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x)f\left(yf(x)-1\right) = x^2f(y) - f(x),$$

for all $x, y \in \mathbb{R}$.

– TST 2

1 For a natural number $k > 1$, define S_k to be the set of all triplets (n, a, b) of natural numbers, with n odd and $\gcd(a, b) = 1$, such that $a + b = k$ and n divides $a^n + b^n$. Find all values of k for which S_k is finite.

2 In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

3 Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

- Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

- Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

Proposed by Warut Suksompong, Thailand

– TST 3

1 Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters $a, b,$ and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

2 Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

3 Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

– TST 4

1 Let ABC be a triangle and AD, BE, CF be cevians concurrent at a point P. Suppose each of the quadrilaterals $PDCE, PEAF$ and $PFBD$ has both circumcircle and incircle. Prove that ABC is equilateral and P coincides with the center of the triangle.

2 Let $n \geq 2$ be a natural number. Let $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n > 0$ and $n(a_1^2 + a_2^2 + \cdots + a_n^2) = 2(a_1 + a_2 + \cdots + a_n)^2$. If $m = \lfloor n/2 \rfloor + 1$, the smallest integer larger than $n/2$, then show that $a_m > 0$.

3 Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$

is divisible by n. Here, we let $a_i = a_{i-n}$ when $i > n$.

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