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– Day 1

1 Consider a regular polygon $A_1A_2\dots A_{6n+3}$. The vertices $A_{2n+1}, A_{4n+2}, A_{6n+3}$ are called *holes*. Initially there are three pebbles in some vertices of the polygon, which are also vertices of equilateral triangle. Players A and B take moves in turn. In each move, starting from A, the player chooses pebble and puts it to the next vertex clockwise (for example, $A_2 \to A_3$, $A_{6n+3} \to A_1$). Player A wins if at least two pebbles lie in holes after someone's move. Does player A always have winning strategy?

Proposed by Bohdan Rublov

2 Find all functions from positive integers to itself such that $f(a+b) = f(a) + f(b) + f(c) + f(d)$ for all $c^2 + d^2 = 2ab$

3 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

– Day 2

4 Find all positive integers a such that for any positive integer $n \ge 5$ we have $2^n - n^2 \mid a^n - n^a$.

5 Let ABC be an equilateral triangle of side 1. There are three grasshoppers sitting in A, B, C. At any point of time for any two grasshoppers separated by a distance d one of them can jump over other one so that distance between them becomes $2kd$, k, d are nonfixed positive integers. Let M, N be points on rays AB, AC such that $AM = AN = l$, l is fixed positive integer. In a finite number of jumps all of grasshoppers end up sitting inside the triangle AMN. Find, in terms of l, the number of final positions of the grasshoppers. (Grasshoppers can leave the triangle AMN during their jumps.)

6 Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1\le r<s\le 2n} (s-r-n)x_rx_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

– Day 3

7 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \dots, n+1$. Prove that if all the numbers $x_1, x_2, \dots, x_{n+1}$ are integers, then $x_1 x_2 \dots x_{n+1} - 1$ is divisible by an odd prime.

8 Let ABC be an acute triangle with $AB < BC$. Let I be the incenter of ABC, and let ω be the circumcircle of ABC. The incircle of ABC is tangent to the side BC at K. The line AK meets ω again at T. Let M be the midpoint of the side BC, and let N be the midpoint of the arc BAC of ω. The segment NT intersects the circumcircle of BIC at P. Prove that $PM \parallel AK$.

9 Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \le n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Proposed by Finland

– Day 4

10 Let $a_1, \dots, a_n$ be real numbers. Define polynomials f, g by

$$f(x) = \sum_{k=1}^{n} a_k x^k, \ g(x) = \sum_{k=1}^{n} \frac{a_k}{2^k - 1} x^k.$$

Assume that $g(2016) = 0$. Prove that $f(x)$ has a root in $(0; 2016)$.

11 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

12 Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.
