IMO Shortlist 2017

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by Snakes, math90, Abbas11235, Tsukuyomi, SHARKYKESA, ShinyDitto, MarkBcc168, Amir Hossein, fattypiggy123, GeronimoStilton, fastlikearabbit, cjquines0, Muradjl

– Algebra

A1 Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

A2 Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

-In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
-In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the first line.
-In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the first line.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

A3 Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

A4 A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.

A5 An integer $n \geq 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

A6 Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

Proposed by Dorlir Ahmeti, Albania

A7 Let $a_0, a_1, a_2, \dots$ be a sequence of integers and $b_0, b_1, b_2, \dots$ be a sequence of *positive* integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1} & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1} & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \dots.$$

for $n = 1, 2, \dots$. Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

A8 A function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

For every $x, y \in \mathbb{R}$ such that $(f(x) + y)(f(y) + x) > 0,$ we have $f(x) + y = f(y) + x.$

Prove that $f(x) + y \leq f(y) + x$ whenever $x > y$.

– Combinatorics

C1 A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Proposed by Jeck Lim, Singapore

C2 Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters $a, b,$ and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

C3 Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

- Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.
- Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

Proposed by Warut Suksompong, Thailand

C4 An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:
(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players, $\vdots$
(N) no one stands between the two shortest players.

Show that this is always possible.

Proposed by Grigory Chelnokov, Russia

C5 A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, $A_0,$ and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:

-The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly $1.$

-A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most $1.$

-The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly $1.$

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Proposed by Gerhard Woeginger, Austria

C6 Let $n>1$ be a given integer. An $n\times n\times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one colour. For each $n\times n\times 1$ box consisting of n^2 unit cubes (in any of the three possible orientations), we consider the set of colours present in that box (each colour is listed only once). This way, we get $3n$ sets of colours, split into three groups according to the orientation.

It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colours that are present.

C7 For any finite sets X and Y of positive integers, denote by $f_X(k)$ the k^{th} smallest positive integer not in X, and let

$$X*Y=X\cup\{f_X(y):y\in Y\}.$$

Let A be a set of $a>0$ positive integers and let B be a set of $b>0$ positive integers. Prove that if $A*B=B*A$, then

$$\underbrace{A*(A*\cdots(A*(A*A))\cdots)}_{\text{A appears } b \text{ times}}=\underbrace{B*(B*\cdots(B*(B*B))\cdots)}_{\text{B appears } a \text{ times}}.$$

Proposed by Alex Zhai, United States

C8 Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The *neighborhood* of a lattice point c consists of all lattice points within the axis-aligned $(2n+1)\times(2n+1)$ square entered at c, apart from c itself. We call a butterfly *lonely*, *crowded*, or *comfortable*, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N. Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

– Geometry

G1 Let $ABCDE$ be a convex pentagon such that $AB=BC=CD$, $\angle EAB=\angle BCD$, and $\angle EDC=\angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

G2 Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Proposed by Charles Leytem, Luxembourg

G3 Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

G4 In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

G5 Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

G6 Let $n \geq 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

G7 A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

G8 There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A tangent segment is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn.

Find all possible numbers of tangent segments when Luciano stops drawing.

– Number Theory

N1 For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Proposed by Stephan Wagner, South Africa

N2 Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i . 10^i$$

.
The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

Proposed by Amine Natik, Morocco

N3 Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$

is divisible by n. Here, we let $a_i = a_{i-n}$ when $i > n$.

Proposed by Warut Suksompong, Thailand

N4 Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is $m-$*tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leq k < t$. Let $S(m)$ be the set of $m-$tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

N5 Find all pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

N6 Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

N7 An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Proposed by John Berman, United States

N8 Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ satisfies the following properties:

- $f(1, 1) = 0$.
- $f(a, b) + f(b, a) = 1$ for any pair of relatively prime positive integers (a, b) not both equal to 1;
- $f(a + b, b) = f(a, b)$ for any pair of relatively prime positive integers (a, b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \sqrt{2p} - 2.$$
