Spain Mathematical Olympiad 2006

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by Snakes, parmenides51

–	Day 1
1	Let $P(x)$ be a polynomial with integer coefficients. Prove that if there is an integer k such that none of the integers $P(1), P(2), ..., P(k)$ is divisible by k, then $P(x)$ does not have integer roots.
2	The dimensions of a wooden octahedron are natural numbers. We painted all its surface (the six faces), cut it by planes parallel to the cubed faces of an edge unit and observed that exactly half of the cubes did not have any painted faces. Prove that the number of octahedra with such property is finite. (It may be useful to keep in mind that $\sqrt[3]{\frac{1}{2}} = 1,79... < 1,8$). Las dimensiones de un ortoedro de madera son enteras. Pintamos toda su superficie (las seis caras), lo cortamos mediante planos paralelos a las caras en cubos de una unidad de arista y observamos que exactamente la mitad de los cubos no tienen ninguna cara pintada. Probar que el número de ortoedros con tal propiedad es finito
3	ABC is an isosceles triangle with $AB = AC$. Let P be any point of a circle tangent to the sides AB in B and to AC in C. Denote a, b and c to the distances from P to the sides BC, AC and AB respectively. Prove that: $a^2 = bc$
–	Day 2
1	Find all the functions $f : (0, +\infty) \rightarrow R$ that satisfy the equation $$f(x)f(y) + f(\frac{\lambda}{x})f(\frac{\lambda}{y}) = 2f(xy)$$ for all pairs of x, y real and positive numbers, where λ is a positive real number such that $f(\lambda) = 1$
2	Prove that the product of four consecutive natural numbers can not be neither square nor perfect cube.
3	The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at E. Denotes by S_1, S_2 and S the areas of the triangles ABE, CDE and the quadrilateral $ABCD$ respectively. Prove that $\sqrt{S_1} + \sqrt{S_2} \leq \sqrt{S}$. When equality is reached?