Tuymaada Olympiad 2018

www.artofproblemsolving.com/community/c686993

by Snakes, ACGNmath

–	Juniors
–	**Day 1**
1	Real numbers $a \neq 0, b, c$ are given. Prove that there is a polynomial $P(x)$ with real coefficients such that the polynomial x^2+1 divides the polynomial $aP(x)^2+bP(x)+c$. *Proposed by A. Golovanov*
2	A circle touches the side AB of the triangle ABC at A, touches the side BC at P and intersects the side AC at Q. The line symmetrical to PQ with respect to AC meets the line AP at X. Prove that $PC=CX$. *Proposed by S. Berlov*
3	n rooks and k pawns are arranged on a 100×100 board. The rooks cannot leap over pawns. For which minimum k is it possible that no rook can capture any other rook? Junior League: $n=2551$ (*Proposed by A. Kuznetsov*) Senior League: $n=2550$ (*Proposed by N. Vlasova*)
4	Prove that for every positive integer $d > 1$ and m the sequence $a_n=2^{2^n}+d$ contains two terms a_k and a_l ($k \neq l$) such that their greatest common divisor is greater than m. *Proposed by T. Hakobyan*
–	**Day 2**
5	99 identical balls lie on a table. 50 balls are made of copper, and 49 balls are made of zinc. The assistant numbered the balls. Once spectrometer test is applied to 2 balls and allows to determine whether they are made of the same metal or not. However, the results of the test can be obtained only the next day. What minimum number of tests is required to determine the material of each ball if all the tests should be performed today? *Proposed by N. Vlasova, S. Berlov*
6	The numbers $1,2,3,\dots,1024$ are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values.

Proposed by A. Golovanov

---

7 Prove the inequality

$$(x^3+2y^2+3z)(4y^3+5z^2+6x)(7z^3+8x^2+9y) \geq 720(xy+yz+xz)$$

for $x, y, z \geq 1$.

Proposed by K. Kokhas

---

8 Quadrilateral $ABCD$ with perpendicular diagonals is inscribed in a circle with centre O. The tangents to this circle at A and C together with line BD form the triangle Δ. Prove that the circumcircles of BOD and Δ are tangent.

Show that this point lies belongs to ω, the circumcircle of OAC

Proposed by A. Kuznetsov

– Seniors

– **Day 1**

---

1 Do there exist three different quadratic trinomials $f(x), g(x), h(x)$ such that the roots of the equation $f(x) = g(x)$ are 1 and 4, the roots of the equation $g(x) = h(x)$ are 2 and 5, and the roots of the equation $h(x) = f(x)$ are 3 and 6?

Proposed by A. Golovanov

---

2 same as juniors Q3

---

3 A point P on the side AB of a triangle ABC and points S and T on the sides AC and BC are such that $AP = AS$ and $BP = BT$. The circumcircle of PST meets the sides AB and BC again at Q and R, respectively. The lines PS and QR meet at L. Prove that the line CL bisects the segment PQ.

Proposed by A. Antropov

---

4 same as juniors Q4

– **Day 2**

---

5 A prime p and a positive integer n are given. The product

$$(1^3+1)(2^3+1)...((n-1)^3+1)(n^3+1)$$

is divisible by p^3. Prove that $p \leq n+1$.

Proposed by Z. Luria

---

6 same as juniors Q7

---

7 A school has three senior classes of M students each. Every student knows at least $\frac{3}{4}M$ people in each of the other two classes. Prove that the school can send M non-intersecting teams to the olympiad so that each team consists of 3 students from different classes who know each other.

Proposed by C. Magyar, R. Martin

---

8 same as juniors Q8
