EGMO 2015

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Day 1 April 16th

1 Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F.
If $\angle ADF = 45^\circ$, show that CF is tangent to ω.

2 A *domino* is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

3 Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \ldots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \ldots, x_m$.

Day 2 April 17th

4 Determine whether there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers which satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every positive integer n.

5 Let m, n be positive integers with $m > 1$. Anastasia partitions the integers $1, 2, \ldots, 2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers.
Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n.

6 Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60$ if and only if $HG = GP'$