South East Mathematical Olympiad 2018

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by Henry_2001, MarkBcc168, mofumofu

–	Grade 10
Day1	July 30th
1	Assume c is a real number. If there exists $x \in [1,2]$ such that $\max\left\{\left\|x+\frac{c}{x}\right\|, \left\|x+\frac{c}{x}+2\right\|\right\} \geq 5$, please find the value range of c.
2	In a Cartesian plane, if both horizontal coordinate and vertical coordinate of a point are rational numbers, we call the point *rational point*. Otherwise, we call it *irrational point*. Consider an arbitrary regular pentagon on the Cartesian plane. Please compare the number of rational point and the number of irrational point among the five vertices of the pentagon. Prove your conclusion.
3	Let O be the circumcenter of acute $\triangle ABC$($AB < AC$), the angle bisector of $\angle BAC$ meets BC at T and M is the midpoint of AT. Point P lies inside $\triangle ABC$ such that $PB \perp PC$. D, E distinct from P lies on the perpendicular to AP through P such that $BD = BP, CE = CP$. If AO bisects segment DE, prove that AO is tangent to the circumcircle of $\triangle AMP$.
4	Does there exist a set $A \subseteq \mathbb{N}^*$ such that for any positive integer n, $A \cap \{n, 2n, 3n, ..., 15n\}$ contains exactly one element? Please prove your conclusion.
Day2	July 31st
5	Let $\{a_n\}$ be a nonnegative real sequence. Define $$X_k = \sum_{i=1}^{2^k} a_i, Y_k = \sum_{i=1}^{2^k} \left\lfloor \frac{2^k}{i} \right\rfloor a_i, k = 0, 1, 2, ...$$ Prove that $X_n \leq Y_n - \sum_{i=0}^{n-1} Y_i \leq \sum_{i=0}^{n} X_i$ for all positive integer n. Here $\lfloor \alpha \rfloor$ denotes the largest integer that does not exceed α.
6	In the isosceles triangle ABC with $AB = AC$, the center of $\odot O$ is the midpoint of the side BC, and AB, AC are tangent to the circle at points E, F respectively. Point G is on $\odot O$ with $\angle AGE = 90^\circ$. A tangent line of $\odot O$ passes through G, and meets AC at K. Prove that line BK bisects EF.
7	There are 24 participants attended a meeting. Each two of them shook hands once or not. A total of 216 handshakes occured in the meeting. For any two participants who have shaken

hands, at most 10 among the rest 22 participants have shaken hands with exactly one of these two persons. Define a *friend circle* to be a group of 3 participants in which each person has shaken hands with the other two. Find the minimum possible value of friend circles.

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8 Given a positive integer m. Let

$$A_l = (4l+1)(4l+2)...(4(5^m+1)l)$$

for any positive integer l. Prove that there exist infinite number of positive integer l which

$$5^{5^m l} \mid A_l \text{ and } 5^{5^m l+1} \nmid A_l$$

and find the minimum value of l satisfying the above condition.

– Grade 11

Day1 July 30th

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1 The same as Grade 10 Problem 2

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2 Suppose that a is real number. Sequence $a_1, a_2, a_3, ...$ satisfies

$$a_1 = a, a_{n+1} = \begin{cases} a_n - \frac{1}{a_n}, & a_n \neq 0 \\ 0, & a_n = 0 \end{cases} (n = 1, 2, 3, ..)$$

Find all possible values of a such that $|a_n| < 1$ for all positive integer n.

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3 Let O be the circumcenter of $\triangle ABC$, where $\angle ABC > 90^\circ$ and M is the midpoint of BC. Point P lies inside $\triangle ABC$ such that $PB \perp PC$. D, E distinct from P lies on the perpendicular to AP through P such that $BD = BP, CE = CP$. If quadrilateral $ADOE$ is a parallelogram, prove that

$$\angle OPE = \angle AMB.$$

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4 Does there exist a set $A \subseteq \mathbb{N}^*$ such that for any positive integer n, $A \cap \{n, 2n, 3n, ..., 15n\}$ contains exactly one element and there exists infinitely many positive integer m such that $\{m, m+2018\} \subset A$? Please prove your conclusion.

Day 2 July 31st

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5 The same as Grade 10 Problem 6

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6 Assume integer $m \geq 2$. There are $3m$ people in a meeting, any two of them either shake hands with each other once or not.We call the meeting "n-interesting", only if there exists $n(n \leq 3m-1)$ people of them, the time everyone of whom shakes hands with other $3m-1$ people is exactly $1, 2, \cdots, n$, respectively. If in any "n-interesting" meeting, there exists 3 people of them who shake hands with each other, find the minimum value of n.

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7 For positive integers $m, n,$ define $f(m,n)$ as the number of ordered triples (x,y,z) of integers such that

$$\begin{cases} xyz = x + y + z + m, \\ \max\{|x|, |y|, |z|\} \leq n \end{cases}$$

Does there exist positive integers $m, n,$ such that $f(m,n) = 2018?$ Please prove your conclusion.

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8 Given a positive real $C \geq 1$ and a sequence $a_1, a_2, a_3, \cdots$ satisfying for any positive integer n, $a_n \geq 0$
and for any real $x \geq 1$,

$$\left| x \lg x - \sum_{k=1}^{[x]} \left[\frac{x}{k} \right] a_k \right| \leq Cx,$$

where $[x]$ is defined as the largest integer that does not exceed x. Prove that for any real $y \geq 1$,

$$\sum_{k=1}^{[y]} a_k < 3Cy.$$
