Mexico National Olympiad 2005

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– Day 1

1 Let O be the center of the circumcircle of an acute triangle ABC, let P be any point inside the segment BC. Suppose the circumcircle of triangle BPO intersects the segment AB at point R and the circumcircle of triangle COP intersects CA at point Q.
(i) Consider the triangle PQR, show that it is similar to triangle ABC and that O is its orthocenter.
(ii) Show that the circumcircles of triangles BPO, COP, PQR have the same radius.

2 Given several matrices of the same size. Given a positive integer N, let's say that a matrix is N-balanced if the entries of the matrix are integers and the difference between any two adjacent entries of the matrix is less than or equal to N.
(i) Show that every $2N$-balanced matrix can be written as a sum of two N-balanced matrices.
(ii) Show that every $3N$-balanced matrix can be written as a sum of three N-balanced matrices.

3 Already the complete problem:

Determine all pairs (a,b) of integers different from 0 for which it is possible to find a positive integer x and an integer y such that x is relatively prime to b and in the following list there is an infinity of integers:
$\rightarrow$ $\frac{a+xy}{b}, \frac{a+xy^2}{b^2}, \frac{a+xy^3}{b^3}, \ldots, \frac{a+xy^n}{b^n}, \ldots$

One idea?

:arrow: **View all the problems from XIX Mexican Mathematical Olympiad** (`http://www.mathlinks.ro/Forum/viewtopic.php?t=61319`)

– Day 2

4 A list of numbers $a_1, a_2, \ldots, a_m$ contains an arithmetic trio a_i, a_j, a_k if $i < j < k$ and $2a_j = a_i + a_k$.
Let n be a positive integer. Show that the numbers $1, 2, 3, \ldots, n$ can be reordered in a list that does not contain arithmetic trios.

5 Let N be an integer greater than 1. A deck has N^3 cards, each card has one of N colors, has one of N figures and has one of N numbers (there are no two identical cards). A collection of

cards of the deck is ”complete” if it has cards of every color, or if it has cards of every figure or of all numbers. How many non-complete collections are there such that, if you add any other card from the deck, the collection becomes complete?

6 Let ABC be a triangle and AD be the angle bisector of $< BAC$, with D on BC. Let E be a point on segment BC such that $BD = EC$. Through E draw l a parallel line to AD and let P be a point in l inside the triangle. Let G be the point where BP intersects AC and F be the point where CP intersects AB. Show $BF = CG$.