Mexico National Olympiad 2002

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– Day 1

1 The numbers 1 to 1024 are written one per square on a 32×32 board, so that the first row is $1, 2, ..., 32$, the second row is $33, 34, ..., 64$ and so on. Then the board is divided into four 16×16 boards and the position of these boards is moved round clockwise, so that

AB goes to DA DC CB

then each of the 16×16 boards is divided into four equal 8×8 parts and each of these is moved around in the same way (within the 16×16 board). Then each of the 8×8 boards is divided into four 4×4 parts and these are moved around, then each 4×4 board is divided into 2×2 parts which are moved around, and finally the squares of each 2×2 part are moved around. What numbers end up on the main diagonal (from the top left to bottom right)?

2 $ABCD$ is a parallelogram. K is the circumcircle of ABD. The lines BC and CD meet K again at E and F. Show that the circumcenter of CEF lies on K.

3 Let n be a positive integer. Does n^2 has more positive divisors of the form $4k + 1$ or of the form $4k - 1$?

– Day 2

4 A domino has two numbers (which may be equal) between 0 and 6, one at each end. The domino may be turned around. There is one domino of each type, so 28 in all. We want to form a chain in the usual way, so that adjacent dominos have the same number at the adjacent ends. Dominos can be added to the chain at either end. We want to form the chain so that after each domino has been added the total of all the numbers is odd. For example, we could place first the domino $(3, 4)$, total $3 + 4 = 7$. Then $(1, 3)$, total $1 + 3 + 3 + 4 = 11$, then $(4, 4)$, total $11 + 4 + 4 = 19$. What is the largest number of dominos that can be placed in this way? How many maximum-length chains are there?

5 A *trio* is a set of three distinct integers such that two of the numbers are divisors or multiples of the third. Which *trio* contained in $\{1, 2, ..., 2002\}$ has the largest possible sum? Find all *trios* with the maximum sum.

6 Let $ABCD$ be a quadrilateral with $\angle DAB = \angle ABC = 90^\circ$. Denote by M the midpoint of the side AB, and assume that $\angle CMD = 90^\circ$. Let K be the foot of the perpendicular from the point M to the line CD. The line AK meets BD at P, and the line BK meets AC at Q. Show that $\angle AKB = 90^\circ$ and $\frac{KP}{PA} + \frac{KQ}{QB} = 1$.