Mexico National Olympiad 2001

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–	Day 1
1	Find all 7-digit numbers which are multiples of 21 and which have each digit 3 or 7.
2	Given some colored balls (at least three different colors) and at least three boxes. The balls are put into the boxes so that no box is empty and we cannot find three balls of different colors which are in three different boxes. Show that there is a box such that all the balls in all the other boxes have the same color.
3	$ABCD$ is a cyclic quadrilateral. M is the midpoint of CD. The diagonals meet at P. The circle through P which touches CD at M meets AC again at R and BD again at Q. The point S on BD is such that $BS = DQ$. The line through S parallel to AB meets AC at T. Show that $AT = RC$.
–	Day 2
4	For positive integers n, m define $f(n, m)$ as follows. Write a list of 2001 numbers a_i, where $a_1 = m$, and a_{k+1} is the residue of $a_k^2 \, mod \, n$ (for $k = 1, 2, ..., 2000$). Then put $f(n, m) = a_1 - a_2 + a_3 - a_4 + a_5 - ... + a_{2001}$. For which $n \geq 5$ can we find m such that $2 \leq m \leq n/2$ and $f(m, n) > 0$?
5	ABC is a triangle with $AB < AC$ and $\angle A = 2\angle C$. D is the point on AC such that $CD = AB$. Let L be the line through B parallel to AC. Let L meet the external bisector of $\angle A$ at M and the line through C parallel to AB at N. Show that $MD = ND$.
6	A collector of rare coins has coins of denominations $1, 2, ..., n$ (several coins for each denomination). He wishes to put the coins into 5 boxes so that: (1) in each box there is at most one coin of each denomination; (2) each box has the same number of coins and the same denomination total; (3) any two boxes contain all the denominations; (4) no denomination is in all 5 boxes. For which n is this possible?