Nordic 1993

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by parmenides51

1 Let F be an increasing real function defined for all $x, 0 \le x \le 1$, satisfying the conditions
(i) $F(\frac{x}{3}) = \frac{F(x)}{2}$.
(ii) $F(1-x) = 1 - F(x)$.
Determine $F(\frac{173}{1993})$ and $F(\frac{1}{13})$.

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2 A hexagon is inscribed in a circle of radius r. Two of the sides of the hexagon have length 1, two have length 2 and two have length 3. Show that r satisfies the equation $2r^3 - 7r - 3 = 0$.

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3 Find all solutions of the system of equations $\begin{cases} s(x) + s(y) = x \\ x + y + s(z) = z \\ s(x) + s(y) + s(z) = y - 4 \end{cases}$
where x, y, and z are positive integers, and $s(x), s(y)$, and $s(z)$ are the numbers of digits in the decimal representations of x, y, and z, respectively.

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4 Denote by $T(n)$ the sum of the digits of the decimal representation of a positive integer n.
a) Find an integer N, for which $T(k \cdot N)$ is even for all $k, 1 \le k \le 1992,$ but $T(1993 \cdot N)$ is odd.
b) Show that no positive integer N exists such that $T(k \cdot N)$ is even for all positive integers k.

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