Cono Sur Olympiad 2010

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by parmenides51, fprosk, Leicich

– Day 1

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1 Pedro must choose two irreducible fractions, each with a positive numerator and denominator such that:

-The sum of the fractions is equal to 2.
-The sum of the numerators of the fractions is equal to 1000.

In how many ways can Pedro do this?

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2 On a line, 44 points are marked and numbered $1, 2, 3, \dots, 44$ from left to right. Various crickets jump around the line. Each starts at point 1, jumping on the marked points and ending up at point 44. In addition, each cricket jumps from a marked point to another marked point with a greater number.
When all the crickets have finished jumping, it turns out that for pair i, j with $1 \leq i < j \leq 44$, there was a cricket that jumped directly from point i to point j, without visiting any of the points in between the two.
Determine the smallest number of crickets such that this is possible.

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3 Let us define *cutting* a convex polygon with n sides by choosing a pair of consecutive sides AB and BC and substituting them by three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, the triangle MBN is removed and a convex polygon with $n + 1$ sides is obtained.
Let P_6 be a regular hexagon with area 1. P_6 is *cut* and the polygon P_7 is obtained. Then P_7 is cut in one of seven ways and polygon P_8 is obtained, and so on. Prove that, regardless of how the cuts are made, the area of P_n is always greater than $2/3$.

– Day 2

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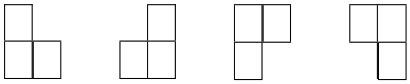
4 Pablo and Silvia play on a 2010×2010 board. To start the game, Pablo writes an integer in every cell. After he is done, Silvia repeats the following operation as many times as she wants: she chooses three cells that form an L, like in the figure below, and adds 1 to each of the numbers in these three cells. Silvia wins if, after doing the operation many times, all of the numbers in the board are multiples of 10.
Prove that Silvia can always win.



5 The incircle of triangle ABC touches sides BC, AC, and AB at D, E, and F respectively. Let ω_a, ω_b and ω_c be the circumcircles of triangles EAF, DBF, and DCE, respectively. The lines DE and DF cut ω_a at $E_a \neq E$ and $F_a \neq F$, respectively. Let r_A be the line E_aF_a. Let r_B and r_C be defined analogously. Show that the lines r_A, r_B, and r_C determine a triangle with its vertices on the sides of triangle ABC.

6 Determine if there exists an infinite sequence $a_0, a_1, a_2, a_3, ...$ of nonegative integers that satisfies the following conditions:
(i) All nonegative integers appear in the sequence exactly once.
(ii) The succession $b_n = a_n + n,$, $n \geq 0$,
is formed by all prime numbers and each one appears exactly once.