European Mathematical Cup 2017

www.artofproblemsolving.com/community/c691109

by parmenides51, Duarti, Ferid.—., BartSimpsons, deimis1231

–	Junior Division
1	Solve in integers the equation : $x^2y+y^2=x^3$
2	A regular hexagon in the plane is called sweet if its area is equal to 1. Is it possible to place 2000000 sweet hexagons in the plane such that the union of their interiors is a convex polygon of area at least 1900000? Remark: A subset S of the plane is called convex if for every pair of points in S, every point on the straight line segment that joins the pair of points also belongs to S. The hexagons may overlap.
3	Let ABC be an acute triangle. Denote by H and M the orthocenter of ABC and the midpoint of side BC, respectively. Let Y be a point on AC such that YH is perpendicular to MH and let Q be a point on BH such that QA is perpendicular to AM. Let J be the second point of intersection of MQ and the circle with diameter MY. Prove that HJ is perpendicular to AM. (Steve Dinh)
4	The real numbers x,y,z satisfy $x^2+y^2+z^2=3$. Prove that the inequality $x^3-(y^2+yz+z^2)x+yz(y+z)\le 3\sqrt{3}$. and find all triples (x,y,z) for which equality holds.
–	Senior Division
1	Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that the inequality $$f(x)+yf(f(x))\le x(1+f(y))$$ holds for all positive integers x,y. Proposed by Adrian Beker.
2	A friendly football match lasts 90 minutes. In this problem, we consider one of the teams, coached by Sir Alex, which plays with 11 players at all times.

a) Sir Alex wants for each of his players to play the same integer number of minutes, but each player has to play less than 60 minutes in total. What is the minimum number of players required?
b) For the number of players found in a), what is the minimum number of substitutions required, so that each player plays the same number of minutes?

Remark: Substitutions can only take place after a positive integer number of minutes, and players who have come off earlier can return to the game as many times as needed. There is no limit to the number of substitutions allowed.

Proposed by Athanasios Kontogeorgis and Demetres Christofides.

---

3 Let ABC be a scalene triangle and let its incircle touch sides BC, CA and AB at points D, E and F respectively. Let line AD intersect this incircle at point X. Point M is chosen on the line FX so that the
quadrilateral $AFEM$ is cyclic. Let lines AM and DE intersect at point L and let Q be the midpoint of segment AE. Point T is given on the line LQ such that the quadrilateral $ALDT$ is cyclic. Let S be a point such that
the quadrilateral $TFSA$ is a parallelogram, and let N be the second point of intersection of the circumcircle of
triangle ASX and the line TS. Prove that the circumcircles of triangles TAN and LSA are tangent to each
other.

---

4 Find all polynomials P with integer coefficients such that $P(0) \neq 0$ and

$$P^n(m) \cdot P^m(n)$$

is a square of an integer for all nonnegative integers n, m.

Remark: For a nonnegative integer k and an integer n, $P^k(n)$ is defined as follows: $P^k(n) = n$ if $k = 0$ and $P^k(n) = P(P(^{k-1}(n))$ if $k > 0$.

Proposed by Adrian Beker.
