All-Russian Olympiad 1994

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–	Grade level 9
–	Day 1
1	Prove that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1$, then $x+y=0$.
2	Two circles S_1 and S_2 touch externally at F. their external common tangent touches S_1 at A and S_2 at B. A line, parallel to AB and tangent to S_2 at C, intersects S_1 at D and E. Prove that points A, F, C are collinear. (A. Kalinin)
3	There are three piles of matches on the table: one with 100 matches, one with 200, and one with 300. Two players play the following game. They play alternatively, and a player on turn removes one of the piles and divides one of the remaining piles into two nonempty piles. The player who cannot make a legal move loses. Who has a winning strategy? (K. Kokhas)
4	On a line are given n blue and n red points. Prove that the sum of distances between pairs of points of the same color does not exceed the sum of distances between pairs of points of different colors. (O. Musin)
–	Day 2
5	Prove the equality $\frac{a_1}{a_2(a_1+a_2)}+\frac{a_2}{a_3(a_2+a_3)}+...+\frac{a_n}{a_1(a_n+a_1)}$ $=\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{a_2(a_2+a_3)}+...+\frac{a_1}{a_n(a_n+a_1)}$ (R. Zhenodarov)
6	Cards numbered with numbers 1 to 1000 are to be placed on the cells of a 1×1994 rectangular board one by one, according to the following rule: If the cell next to the cell containing the card n is free, then the card $n+1$ must be put on it. Prove that the number of possible arrangements is not more than half a mllion.

7 A trapezoid $ABCD$ ($AB///CD$) has the property that there are points P and Q on sides AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that the points P and Q are equidistant from the intersection point of the diagonals of the trapezoid.

(M. Smurov)

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8 A plane is divided into unit squares by two collections of parallel lines. For any $n \times n$ square with sides on the division lines, we define its frame as the set of those unit squares which internally touch the boundary of the $n \times n$ square. Prove that there exists only one way of covering a given 100×100 square whose sides are on the division lines with frames of 50 squares (not necessarily contained in the 100×100 square).

(A. Perlin)

– Grade level 10

– Day 1

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1 Let be given three quadratic polynomials: $P_1(x) = x^2 + p_1x + q_1, P_2(x) = x^2 + p_2x + q_2, P_3(x) = x^2 + p_3x + q_3$.
Prove that the equation $|P_1(x)| + |P_2(x)| = |P_3(x)|$ has at most eight real roots.

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2 same as grade 9 P3

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3 Let a, b, c be the sides of a triangle, let m_a, m_b, m_c be the corresponding medians, and let D be the diameter of the circumcircle of the triangle.
Prove that $\frac{a^2+b^2}{m_c} + \frac{a^2+c^2}{m_b} + \frac{b^2+c^2}{m_a} \leq 6D$.

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4 In a regular $6n + 1$-gon, k vertices are painted in red and the others in blue. Prove that the number of isosceles triangles whose vertices are of the same color does not depend on the arrangement of the red vertices.

– Day 2

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5 Prove that, for any natural numbers k, m, n: $[k, m] \cdot [m, n] \cdot [n, k] \geq [k, m, n]^2$

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6 I'll post some nice combinatorics problems here, taken from the wonderful training book "Les olympiades de mathmatiques" (in French) written by Tarik Belhaj Soulami.
Here goes the first one:

Let $\mathbb{I}$ be a non-empty subset of $\mathbb{Z}$ and let f and g be two functions defined on $\mathbb{I}$. Let m be the number of pairs $(x,\ y)$ for which $f(x) = g(y)$, let n be the number of pairs $(x,\ y)$ for which

$f(x) = f(y)$ and let k be the number of pairs $(x,\ y)$ for which $g(x) = g(y)$. Show that

$$2m \leq n + k.$$

7 Let Γ_1, Γ_2 and Γ_3 be three non-intersecting circles,which are tangent to the circle Γ at points A_1, B_1, C_1,respectively.Suppose that common tangent lines to (Γ_2, Γ_3),(Γ_1, Γ_3),(Γ_2, Γ_1) intersect in points A, B, C.
Prove that lines AA_1, BB_1, CC_1 are concurrent.

8 There are 30 students in a class. In an examination, their results were all different from each other. It is given that everyone has the same number of friends. Find the maximum number of students such that each one of them has a better result than the majority of his friends.
PS. Here majority means larger than half.

– Grade level 11

– Day 1

1 All Russian MO 1994 Grade 11
"Natural numbers a and b are such that $\frac{a+1}{b} + \frac{b+1}{a}$ is an integer. If d is the greatest common divisor of a and b, prove that $d^2 \leq a + b$.

2 Inside a convex 100-gon are selected k points, $2 \leq k \leq 50$. Show that one can choose $2k$ vertices of the 100-gon so that the convex $2k$-gon determined by these vertices contains all the selected points.

3 Two circles S_1 and S_2 touch externally at F. their external common tangent touches S_1 at A and S_2 at B. A line, parallel to AB and tangent to S_2 at C, intersects S_1 at D and E. Prove that the common chord of the circumcircles of triangles ABC and BDE passes through point F.

(A. Kalinin)

4 Real numbers are written on the squares of an infinite grid. Two figures consisting of finitely many squares are given. They may be translated anywhere on the grid as long as their squares coincide with those of the grid. It is known that wherever the first figure is translated, the sum of numbers it covers is positive. Prove that the second figure can be translated so that the sum of the numbers it covers is also positive.

– Day 2

5 Let a_1 be a natural number not divisible by 5. The sequence $a_1, a_2, a_3, ...$ is defined by $a_{n+1} = a_n + b_n$, where b_n is the last digit of a_n. Prove that the sequence contains infinitely many powers of two.

(N. Agakhanov)

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6 same as grade 10 P6

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7 The altitudes AA_1, BB_1, CC_1, DD_1 of a tetrahedron $ABCD$ intersect in the center H of the sphere inscribed in the tetrahedron $A_1B_1C_1D_1$. Prove that the tetrahedron $ABCD$ is regular.

(D. Tereshin)

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8 Players A, B alternately move a knight on a 1994×1994 chessboard. Player A makes only horizontal moves, i.e. such that the knight is moved to a neighboring row, while B makes only vertical moves. Initally player A places the knight to an arbitrary square and makes the first move. The knight cannot be moved to a square that was already visited during the game. A player who cannot make a move loses. Prove that player A has a winning strategy.
