JBMO Shortlist 2017

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by parmenides51, totode, sqing

–	Algebra
A1	Let a,b,c be positive real numbers such that $a+b+c+ab+bc+ca+abc=7$. Prove that $\sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2}\geq 6$.
A2	Let a and b be positive real numbers such that $3a^2+2b^2=3a+2b$. Find the minimum value of $A=\sqrt{\frac{a}{b(3a+2)}}+\sqrt{\frac{b}{a(2b+3)}}$
A3	let $a\leq b\leq c\leq d$ show that: $$ab^3+bc^3+cd^3+da^3\geq a^2b^2+b^2c^2+c^2d^2+d^2a^2$$
A4	Let x,y,z be positive integers such that $x\neq y\neq z\neq x$ .Prove that $$(x+y+z)(xy+yz+zx-2)\geq 9xyz.$$ When does the equality hold? *Proposed by Dorlir Ahmeti, Albania*
–	Combinatorics
C1	Consider a regular $2n+1$-gon P in the plane, where n is a positive integer. We say that a point S on one of the sides of P can be seen from a point E that is external to P, if the line segment SE contains no other points that lie on the sides of P except S. We want to color the sides of P in 3 colors, such that every side is colored in exactly one color, and each color must be used at least once. Moreover, from every point in the plane external to P, at most 2 different colors on P can be seen (ignore the vertices of P, we consider them colorless). Find the largest positive integer for which such a coloring is possible.
C2	Consider a regular 2n-gon P,$A_1,A_2,\cdots ,A_{2n}$ in the plane ,where n is a positive integer . We say that a point S on one of the sides of P can be seen from a point E that is external to P , if the line segment SE contains no other points that lie on the sides of P except S .We color the sides of P in 3 different colors (ignore the vertices of P,we consider them colorless), such that every side is colored in exactly one color, and each color is used at least once . Moreover ,from every point in the plane external to P , points of most 2 different colors on P can be seen .Find

the number of distinct such colorings of P (two colorings are considered distinct if at least one of sides is colored differently).

Proposed by Viktor Simjanoski, Macedonia

JBMO 2017, Q4

C3 We have two piles with 2000 and 2017 coins respectively.
Ann and Bob take alternate turns making the following moves:
The player whose turn is to move picks a pile with at least two coins, removes from that pile t coins for some $2 \leq t \leq 4$, and adds to the other pile 1 coin. The players can choose a different t at each turn, and the player who cannot make a move loses.
If Ann plays first determine which player has a winning strategy.

– Geometry

G1 Given a parallelogram $ABCD$. The line perpendicular to AC passing through C and the line perpendicular to BD passing through A intersect at point P. The circle centered at point P and radius PC intersects the line BC at point X, ($X \neq C$) and the line DC at point Y , ($Y \neq C$). Prove that the line AX passes through the point Y .

G2 Let ABC be an acute triangle such that AB is the shortest side of the triangle. Let D be the midpoint of the side AB and P be an interior point of the triangle such that $\angle CAP = \angle CBP = \angle ACB$. Denote by M and N the feet of the perpendiculars from P to BC and AC, respectively. Let p be the line through M parallel to AC and q be the line through N parallel to BC. If p and q intersect at K prove that D is the circumcenter of triangle MNK.

G3 Consider triangle ABC such that $AB \leq AC$. Point D on the arc BC of thecircumcirle of ABC not containing point A and point E on side BC are such that $\angle BAD = \angle CAE < \frac{1}{2}\angle BAC$. Let S be the midpoint of segment AD. If $\angle ADE = \angle ABC - \angle ACB$ prove that $\angle BSC = 2\angle BAC$.

G4 Let ABC be an acute triangle such that $AB \neq AC$,with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that $\angle BQM = \angle BCA$ and $\angle CQM = \angle CBA$. Let the line AO intersect Γ at E $(E \neq A)$ and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the point A, M and X are collinear.

G5 A point P lies in the interior of the triangle ABC. The lines AP, BP, and CP intersect BC, CA, and AB at points D, E, and F, respectively. Prove that if two of the quadrilaterals $ABDE, BCEF, CAFD, A$ and $CDPE$ are concyclic, then all six are concyclic.

– Number Theory

NT1 Determine all the sets of six consecutive positive integers such that the product of some two of them . added to the product of some other two of them is equal to the product of the remaining two numbers.

NT2 Determine all positive integers n such that $n^2/(n-1)!$

NT3 Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square.

NT4 Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

NT5 Find all positive integers n such that there exists a prime number p, such that $p^n - (p-1)^n$ is a power of 3.

Note. A power of 3 is a number of the form 3^a where a is a positive integer.
