

Brazil National Olympiad 1981

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by parmenides51

- 1 For which k does the system $x^2 - y^2 = 0$, $(x - k)^2 + y^2 = 1$ have exactly:
(i) two,
(ii) three real solutions?

- 2 Show that there are at least 3 and at most 4 powers of 2 with m digits. For which m are there 4?

- 3 Given a sheet of paper and the use of a rule, compass and pencil, show how to draw a straight line that passes through two given points, if the length of the ruler and the maximum opening of the compass are both less than half the distance between the two points. You may not fold the paper.

- 4 A graph has 100 points. Given any four points, there is one joined to the other three. Show that one point must be joined to all 99 other points. What is the smallest number possible of such points (that are joined to all the others)?

- 5 Two thieves stole a container of 8 liters of wine. How can they divide it into two parts of 4 liters each if all they have is a 3 liter container and a 5 liter container? Consider the general case of dividing $m + n$ liters into two equal amounts, given a container of m liters and a container of n liters (where m and n are positive integers). Show that it is possible iff $m + n$ is even and $(m + n)/2$ is divisible by $\gcd(m, n)$.

- 6 The centers of the faces of a cube form a regular octahedron of volume V . Through each vertex of the cube we may take the plane perpendicular to the long diagonal from the vertex. These planes also form a regular octahedron. Show that its volume is $27V$.
