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by parmenides51

1 $p(x_1, x_2, ..., x_n)$ is a polynomial with integer coefficients. For each positive integer $r, k(r)$ is the number of n-tuples $(a_1, a_2, ..., a_n)$ such that $0 \le a_i \le r-1$ and $p(a_1, a_2, ..., a_n)$ is prime to r. Show that if u and v are coprime then $k(u \cdot v) = k(u) \cdot k(v)$, and if p is prime then $k(p^s) = p^{n(s-1)} k(p)$.

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2 Given a point p inside a convex polyhedron P. Show that there is a face F of P such that the foot of the perpendicular from p to F lies in the interior of F.

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3 Two players play alternately. The first player is given a pair of positive integers (x_1, y_1). Each player must replace the pair (x_n, y_n) that he is given by a pair of non-negative integers (x_{n+1}, y_{n+1}) such that $x_{n+1} = min(x_n, y_n)$ and $y_{n+1} = max(x_n, y_n) - k \cdot x_{n+1}$ for some positive integer k. The first player to pass on a pair with $y_{n+1} = 0$ wins. Find for which values of x_1/y_1 the first player has a winning strategy.

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4 Given points $A_1(x_1, y_1, z_1), A_2(x_2, y_2, z_2), .., A_n(x_n, y_n, z_n)$ let $P(x, y, z)$ be the point which minimizes $\Sigma(|x - x_i| + |y - y_i| + |z - z_i|)$. Give an example (for each $n > 4$) of points A_i for which the point P lies outside the convex hull of the points A_i.

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5 A and B wish to divide a cake into two pieces. Each wants the largest piece he can get. The cake is a triangular prism with the triangular faces horizontal. A chooses a point P on the top face. B then chooses a vertical plane through the point P to divide the cake. B chooses which piece to take. Which point P should A choose in order to secure as large a slice as possible?
