South Africa National Olympiad 2018

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by DylanN

1 One hundred empty glasses are arranged in a 10×10 array. Now we pick a of the rows and pour blue liquid into all glasses in these rows, so that they are half full. The remaining rows are filled halfway with yellow liquid. Afterwards, we pick b of the columns and fill them up with blue liquid. The remaining columns are filled up with yellow liquid. The mixture of blue and yellow liquid turns green. If both halves have the same colour, then that colour remains as it is.

- Determine all possible combinations of values for a and b so that exactly half of the glasses contain green liquid at the end.
- Is it possible that precisely one quarter of the glasses contain green liquid at the end?

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2 In a triangle ABC, $AB = AC$, and D is on BC. A point E is chosen on AC, and a point F is chosen on AB, such that $DE = DC$ and $DF = DB$. It is given that $\frac{DC}{BD} = 2$ and $\frac{AF}{AE} = 5$. Determine that value of $\frac{AB}{BC}$.

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3 Determine the smallest positive integer n whose prime factors are all greater than 18, and that can be expressed as $n = a^3 + b^3$ with positive integers a and b.

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4 Let ABC be a triangle with circumradius R, and let ℓ_A, ℓ_B, ℓ_C be the altitudes through A, B, C respectively. The altitudes meet at H. Let P be an arbitrary point in the same plane as ABC. The feet of the perpendicular lines through P onto ℓ_A, ℓ_B, ℓ_C are D, E, F respectively. Prove that the areas of DEF and ABC satisfy the following equation:

$$\text{area}(DEF) = \frac{PH^2}{4R^2} \cdot \text{area}(ABC).$$

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5 Determine all sequences $a_1, a_2, a_3, \dots$ of nonnegative integers such that $a_1 < a_2 < a_3 < \dots$ and a_n divides $a_{n-1} + n$ for all $n \geq 2$.

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6 Let n be a positive integer, and let $x_1, x_2, \dots, x_n$ be distinct positive integers with $x_1 = 1$. Construct an $n \times 3$ table where the entries of the k-th row are $x_k, 2x_k, 3x_k$ for $k = 1, 2, \dots, n$. Now follow a procedure where, in each step, two identical entries are removed from the table. This continues until there are no more identical entries in the table.

- Prove that at least three entries remain at the end of the procedure.
- Prove that there are infinitely many possible choices for n and $x_1, x_2, \dots, x_n$ such that only three entries remain.

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