Mexico National Olympiad 1996

www.artofproblemsolving.com/community/c691183

by parmenides51

–	Day 1
1	Let P and Q be the points on the diagonal BD of a quadrilateral $ABCD$ such that $BP = PQ = QD$. Let AP and BC meet at E, and let AQ meet DC at F. (a) Prove that if $ABCD$ is a parallelogram, then E and F are the midpoints of the corresponding sides. (b) Prove the converse of (a).
2	There are 64 booths around a circular table and on each one there is a chip. The chips and the corresponding booths are numbered 1 to 64 in this order. At the center of the table there are 1996 light bulbs which are all turned off. Every minute the chips move simultaneously in a circular way (following the numbering sense) as follows: chip 1 moves one booth, chip 2 moves two booths, etc., so that more than one chip can be in the same booth. At any minute, for each chip sharing a booth with chip 1 a bulb is lit. Where is chip 1 on the first minute in which all bulbs are lit?
3	Prove that it is not possible to cover a 6×6 square board with eighteen 2×1 rectangles, in such a way that each of the lines going along the interior gridlines cuts at least one of the rectangles. Show also that it is possible to cover a 6×5 rectangle with fifteen 2×1 rectangles so that the above condition is fulfilled.
–	Day 2
4	For which integers $n \geq 2$ can the numbers 1 to 16 be written each in one square of a squared 4×4 paper such that the 8 sums of the numbers in rows and columns are all different and divisible by n?
5	The numbers 1 to n^2 are written in an n×n squared paper in the usual ordering. Any sequence of right and downwards steps from a square to an adjacent one (by side) starting at square 1 and ending at square n^2 is called a path. Denote by $L(C)$ the sum of the numbers through which path C goes. (a) For a fixed n, let M and m be the largest and smallest $L(C)$ possible. Prove that $M - m$ is a perfect cube. (b) Prove that for no n can one find a path C with $L(C) = 1996$.
6	In a triangle ABC with $AB < BC < AC$, points A', B', C' are such that $AA' \perp BC$ and $AA' = BC, BB' \perp CA$ and $BB' = CA$, and $CC' \perp AB$ and $CC' = AB$, as shown on the picture. Suppose that $\angle AC'B$ is a right angle. Prove that the points A', B', C' are collinear.

