Mexico National Olympiad 1991

www.artofproblemsolving.com/community/c691190

by parmenides51

–	Day 1
1	Evaluate the sum of all positive irreducible fractions less than 1 and having the denominator 1991.
2	A company of n soldiers is such that (i) n is a palindrome number (read equally in both directions); (ii) if the soldiers arrange in rows of $3, 4$ or 5 soldiers, then the last row contains $2, 3$ and 5 soldiers, respectively. Find the smallest n satisfying these conditions and prove that there are infinitely many such numbers n.
3	Four balls of radius 1 are placed in space so that each of them touches the other three. What is the radius of the smallest sphere containing all of them?
–	Day 2
4	The diagonals AC and BD of a convex quarilateral $ABCD$ are orthogonal. Let M, N, R, S be the midpoints of the sides AB, BC, CD and DA respectively, and let W, X, Y, Z be the projections of the points M, N, R and S on the lines CD, DA, AB and BC, respectively. Prove that the points M, N, R, S, W, X, Y and Z lie on a circle.
5	The sum of squares of two consecutive integers can be a square, as in $3^2 + 4^2 = 5^2$. Prove that the sum of squares of m consecutive integers cannot be a square for $m = 3$ or 6 and find an example of 11 consecutive integers the sum of whose squares is a square.
6	Given an n-gon ($n \geq 4$), consider a set T of triangles formed by vertices of the polygon having the following property: Every two triangles in T have either two common vertices, or none. Prove that T contains at most n triangles.