Mexico National Olympiad 1992

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–	Day 1
1	The tetrahedron $OPQR$ has the $\angle POQ = \angle POR = \angle QOR = 90^o$. X, Y, Z are the midpoints of PQ, QR and RP. Show that the four faces of the tetrahedron $OXYZ$ have equal area.
2	Given a prime number p, how many 4-tuples (a, b, c, d) of positive integers with $0 \le a, b, c, d \le p-1$ satisfy $ad = bc$ mod p?
3	Given 7 points inside or on a regular hexagon, show that three of them form a triangle with area $\le 1/6$ the area of the hexagon.
–	Day 2
4	Show that $1 + 11^{11} + 111^{111} + 1111^{1111} + ... + 1111111111^{1111111111}$ is divisible by 100.
5	x, y, z are positive reals with sum 3. Show that $$6 < \sqrt{2x+3} + \sqrt{2y+3} + \sqrt{2z+3} \le 3\sqrt{5}$$
6	$ABCD$ is a rectangle. I is the midpoint of CD. BI meets AC at M. Show that the line DM passes through the midpoint of BC. E is a point outside the rectangle such that $AE = BE$ and $\angle AEB = 90^o$. If $BE = BC = x$, show that EM bisects $\angle AMB$. Find the area of $AEBM$ in terms of x.