2018 China Northern Math Olympiad

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–	Grade 10
–	Test 1
1	In triangle ABC, let the circumcenter, incenter, and orthocenter be O, I, and H respectively. Segments AO, AI, and AH intersect the circumcircle of triangle ABC at D, E, and F. CD intersects AE at M and CE intersects AF at N. Prove that MN is parallel to BC.
2	Let a,b,c be nonnegative reals such that $$a^2+b^2+c^2+ab+\frac{2}{3}ac+\frac{4}{3}bc=1$$ Find the maximum and minimum value of $a+b+c$.
3	Let p be a prime such that $3\|p+1$. Show that $p\|a-b$ if and only if $p\|a^3-b^3$
4	In each square of a 4 by 4 grid, you put either a $+1$ or a -1. If any 2 rows and 2 columns are deleted, the sum of the remaining 4 numbers is nonnegative. What is the minimum number of $+1$'s needed to be placed to be able to satisfy the conditions
–	Test 2
5	A right triangle has the property that it's sides are pairwise relatively prime positive integers and that the ratio of it's area to it's perimeter is a perfect square. Find the minimum possible area of this triangle.
6	Let H be the orthocenter of triangle ABC. Let D and E be points on AB and AC such that DE is parallel to CH. If the circumcircle of triangle BDH passes through M, the midpoint of DE, then prove that $\angle ABM=\angle ACM$
7	If a,b,c are positive reals, prove that $$\frac{a+bc}{a+a^2}+\frac{b+ca}{b+b^2}+\frac{c+ab}{c+c^2}\geq 3$$

8 2 players A and B play the following game with A going first: On each player's turn, he puts a number from 1 to 99 among 99 equally spaced points on a circle (numbers cannot be repeated), and the player who completes 3 consecutive numbers that forms an arithmetic sequence around the circle wins the game. Who has the winning strategy? Explain your reasoning.

– Grade 11

– Test 1

1 The same as Grade 10 Problem 2

2 Let p be a prime. We say p is *good* if and only if for any positive integer $a, b,$ such that

$$a \equiv b(\text{mod} p) \Leftrightarrow a^3 \equiv b^3(\text{mod} p).$$

Prove that
(1)There are infinite primes p which are *good*;
(2)There are infinite primes p which are not *good*.

3 A, B, C, D, E lie on $\odot O$ in that order,and

$$BD \cap CE = F, CE \cap AD = G, AD \cap BE = H, BE \cap AC = I, AC \cap BD = J.$$

Prove that $\frac{FG}{CE} = \frac{GH}{DA} = \frac{HI}{BE} = \frac{IJ}{AC} = \frac{JF}{BD}$ when and only when F, G, H, I, J are concyclic.

4 For $n(n \geq 3)$ positive intengers $a_1, a_2, \cdots, a_n$. Put the numbers on a circle. In each operation, calculate difference between two adjacent numbers and take its absolute value. Put the n numbers we get on another ciecle (do not change their order). Find all n, satisfying that no matter how $a_1, a_2, \cdots, a_n$ are given, all numbers on the circle are equal after limited operations.

– Test 2

5 The same as Grade 10 Problem 6

6 For $a_1 = 3$, define the sequence $a_1, a_2, a_3, \ldots$ for $n \geq 1$ as

$$na_{n+1} = 2(n+1)a_n - n - 2.$$

Prove that for any odd prime p, there exist positive integer m, such that $p|a_m$ and $p|a_{m+1}$.

7 The same as Grade 10 Problem 8

8 Prove that there exist infinite positive integer n, such that $2018|\left(1 + 2^n + 3^n + 4^n\right)$.