Sharygin Geometry Olympiad 2014

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– First (Correspondence) Round

1 A right-angled triangle ABC is given. Its catheus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypotenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

2 A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1 and all angles divisible by 45°?

3 Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, and F are collinear.

4 A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex. Prove that the incenter of the triangle lies inside the square.

5 In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

6 Given a circle with center O and a point P not lying on it, let X be an arbitrary point on this circle and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

7 A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

8 Let $ABCD$ be a rectangle. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

9 Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

10 Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1 and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1 and l_2 and the sides of the angle touch some fixed circle.

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11 Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

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12 Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

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13 Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1, the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

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14 In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

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15 Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.
a) Which of the sidelengths of the triangle is medial (intermediate in length)?
b) Which of the lengths of segments AK, BK, CK is medial (intermediate in length)?

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16 Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC, AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

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17 Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

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18 Let I be the incenter of a circumscribed quadrilateral $ABCD$. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

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19 Two circles ω_1 and ω_2 touch externally at point P.Let A be a point on ω_2 not lying on the line through the centres of the two circles.Let AB and AC be the tangents to ω_1.Lines BP and CP meet ω_2 for the second time at points E and F.Prove that the line EF,the tangent to ω_2 at A and the common tangent at P concur.

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20 A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then
a) the circumcircles of triangles $KD_1A_1, LA_1B_1, MB_1C_1$ and $NC1D1$ also concur at some point Q;
b) point O lies on the perpendicular bisector to PQ.

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21 Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches the sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that EF is parallel to E_1F_1.

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22 Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

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23 Let A, B, C and D be a triharmonic quadruple of points, i.e $AB \cdot CD = AC \cdot BD = AD \cdot BC$. Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that
a) A, B, C_1, D_1 are concyclic;
b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

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24 A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

– Final Round

– grade 8

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1 The incircle of a right-angled triangle ABC touches its catheti AC and BC at points B_1 and A_1, the hypotenuse touches the incircle at point C_1. Lines C_1A_1 and C_1B_1 meet CA and CB respectively at points B_0 and A_0. Prove that $AB_0 = BA_0$.

(J. Zajtseva, D. Shvetsov)

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2 Let AH_a and BH_b be altitudes, AL_a and BL_b be angle bisectors of a triangle ABC. It is known that $H_aH_b//L_aL_b$. Is it necessarily true that $AC = BC$?

(B. Frenkin)

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3 Points M and N are the midpoints of sides AC and BC of a triangle ABC. It is known that $\angle MAN = 15^o$ and $\angle BAN = 45^o$. Find the value of angle $\angle ABM$.

(A. Blinkov)

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4 Tanya has cut out a triangle from checkered paper as shown in the picture. The lines of the grid have faded. Can Tanya restore them without any instruments only folding the triangle (she remembers the triangle sidelengths)?

(T. Kazitsyna)

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5 A triangle with angles of $30, 70$ and 80 degrees is given. Cut it by a straight line into two triangles in such a way that an angle bisector in one of these triangles and a median in the other one drawn from two endpoints of the cutting segment are parallel to each other. (It suffices to find one such cutting.)

(A. Shapovalov)

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6 Two circles k_1 and k_2 with centers O_1 and O_2 are tangent to each other externally at point O. Points X and Y on k_1 and k_2 respectively are such that rays O_1X and O_2Y are parallel and codirectional. Prove that two tangents from X to k_2 and two tangents from Y to k_1 touch the same circle passing through O.

(V. Yasinsky)

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7 Two points on a circle are joined by a broken line shorter than the diameter of the circle. Prove that there exists a diameter which does not intersect this broken line.

(Folklor)

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8 Let M be the midpoint of the chord AB of a circle centered at O. Point K is symmetric to M with respect to O, and point P is chosen arbitrarily on the circle. Let Q be the intersection of the line perpendicular to AB through A and the line perpendicular to PK through P. Let H be the projection of P onto AB. Prove that QB bisects PH.

(Tran Quang Hung)

– grade 9

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1 Let $ABCD$ be a cyclic quadrilateral. Prove that $AC > BD$ if and only if $(AD - BC)(AB - CD) > 0$.

(V. Yasinsky)

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2 In a quadrilateral $ABCD$ angles A and C are right. Two circles with diameters AB and CD meet at points X and Y . Prove that line XY passes through the midpoint of AC.

(F. Nilov)

3 An acute angle A and a point E inside it are given. Construct points B, C on the sides of the angle such that E is the center of the Euler circle of triangle ABC.

(E. Diomidov)

4 Let H be the orthocenter of a triangle ABC. Given that H lies on the incircle of ABC , prove that three circles with centers A, B, C and radii AH, BH, CH have a common tangent.

(Mahdi Etesami Fard)

5 In triangle ABC $\angle B = 60^o, O$ is the circumcenter, and L is the foot of an angle bisector of angle B. The circumcirle of triangle BOL meets the circumcircle of ABC at point $D \neq B$. Prove that $BD \perp AC$.

(D. Shvetsov)

6 Let I be the incenter of triangle ABC, and M, N be the midpoints of arcs ABC and BAC of its circumcircle. Prove that points M, I, N are collinear if and only if$AC + BC = 3AB$.

(A. Polyansky)

7 Nine circles are drawn around an arbitrary triangle as in the figure. All circles tangent to the same side of the triangle have equal radii. Three lines are drawn, each one connecting one of the triangle's vertices to the center of one of the circles touching the opposite side, as in the figure. Show that the three lines are concurrent.

(N. Beluhov)

8 A convex polygon P lies on a flat wooden table. You are allowed to drive some nails into the table. The nails must not go through P, but they may touch its boundary. We say that a set of nails blocks P if the nails make it impossible to move P without lifting it off the table. What is the minimum number of nails that suffices to block any convex polygon P?

(N. Beluhov, S. Gerdgikov)

– grade 10

1 The vertices and the circumcenter of an isosceles triangle lie on four different sides of a square. Find the angles of this triangle.

(I. Bogdanov, B. Frenkin)

2 A circle, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points

X and Y respectively. Lines WX and WY meet AB at points N and M respectively. Prove that the length of segment NM does not depend on point C.

(A. Zertsalov, D. Skrobot)

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3 Do there exist convex polyhedra with an arbitrary number of diagonals (a diagonal is a segment joining two vertices of a polyhedron and not lying on the surface of this polyhedron)?

(A. Blinkov)

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4 Let ABC be a fixed triangle in the plane. Let D be an arbitrary point in the plane. The circle with center D, passing through A, meets AB and AC again at points A_b and A_c respectively. Points B_a, B_c, C_a and C_b are defined similarly. A point D is called good if the points A_b, A_c, B_a, B_c, C_a, and C_b are concyclic. For a given triangle ABC, how many good points can there be?

(A. Garkavyj, A. Sokolov)

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5 The altitude from one vertex of a triangle, the bisector from the another one and the median from the remaining vertex were drawn, the common points of these three lines were marked, and after this everything was erased except three marked points. Restore the triangle. (For every two erased segments, it is known which of the three points was their intersection point.)

(A. Zaslavsky)

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6 The incircle of a non-isosceles triangle ABC touches AB at point C'. The circle with diameter BC' meets the incircle and the bisector of angle B again at points A_1 and A_2 respectively. The circle with diameter AC' meets the incircle and the bisector of angle A again at points B_1 and B_2 respectively. Prove that lines AB, A_1B_1, A_2B_2 concur.

(E. H. Garsia)

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7 Prove that the smallest dihedral angle between faces of an arbitrary tetrahedron is not greater than the dihedral angle between faces of a regular tetrahedron.

(S. Shosman, O. Ogievetsky)

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8 Given is a cyclic quadrilateral $ABCD$. The point L_a lies in the interior of BCD and is such that its distances to the sides of this triangle are proportional to the lengths of corresponding sides. The points L_b, L_c, and L_d are defined analogously. Given that the quadrilateral $L_aL_bL_cL_d$ is cyclic, prove that the quadrilateral $ABCD$ has two parallel sides.

(N. Beluhov)
