Sharygin Geometry Olympiad 2015

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–	Final Round	
–	Grade 8	
1	In trapezoid $ABCD$ angles A and B are right, $AB = AD, CD = BC + AD, BC < AD$. Prove that $\angle ADC = 2\angle ABE$, where E is the midpoint of segment AD. (V. Yasinsky)	
2	A circle passing through A, B and the orthocenter of triangle ABC meets sides AC, BC at their inner points. Prove that $60^o < \angle C < 90^o$. (A. Blinkov)	
3	In triangle ABC we have $AB = BC, \angle B = 20^o$. Point M on AC is such that $AM : MC = 1 : 2$, point H is the projection of C to BM. Find angle AHB. (M. Yevdokimov)	
4	Prove that an arbitrary convex quadrilateral can be divided into five polygons having symmetry axes. (N. Belukhov)	
5	Two equal hard triangles are given. One of their angles is equal to α (these angles are marked). Dispose these triangles on the plane in such a way that the angle formed by some three vertices would be equal to $\alpha/2$. *(No instruments are allowed, even a pencil.)* (E. Bakayev, A. Zaslavsky)	
6	Lines b and c passing through vertices B and C of triangle ABC are perpendicular to sideline BC. The perpendicular bisectors to AC and AB meet b and c at points P and Q respectively. Prove that line PQ is perpendicular to median AM of triangle ABC. (D. Prokopenko)	
7	Point M on side AB of quadrilateral $ABCD$ is such that quadrilaterals $AMCD$ and $BMDC$ are circumscribed around circles centered at O_1 and O_2 respectively. Line O_1O_2 cuts an isosceles triangle with vertex M from angle CMD. Prove that $ABCD$ is a cyclic quadrilateral.	

(M. Kungozhin)

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8 Points C_1, B_1 on sides AB, AC respectively of triangle ABC are such that $BB_1 \perp CC_1$. Point X lying inside the triangle is such that $\angle XBC = \angle B_1BA, \angle XCB = \angle C_1CA$. Prove that $\angle B_1XC_1 = 90^o - \angle A$.

(A. Antropov, A. Yakubov)

– Grade 9

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1 Circles α and β pass through point C. The tangent to α at this point meets β at point B, and the tangent to β at C meets α at point A so that A and B are distinct from C and angle ACB is obtuse. Line AB meets α and β for the second time at points N and M respectively. Prove that $2MN < AB$.

(D. Mukhin)

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2 A convex quadrilateral is given. Using a compass and a ruler construct a point such that its projections to the sidelines of this quadrilateral are the vertices of a parallelogram.

(A. Zaslavsky)

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3 Let 100 discs lie on the plane in such a way that each two of them have a common point. Prove that there exists a point lying inside at least 15 of these discs.

(M. Kharitonov, A. Polyansky)

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4 A fixed triangle ABC is given. Point P moves on its circumcircle so that segments BC and AP intersect. Line AP divides triangle BPC into two triangles with incenters I_1 and I_2. Line I_1I_2 meets BC at point Z. Prove that all lines ZP pass through a fixed point.

(R. Krutovsky, A. Yakubov)

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5 Let BM be a median of nonisosceles right-angled triangle ABC ($\angle B = 90^o$), and Ha, Hc be the orthocenters of triangles ABM, CBM respectively. Prove that lines AH_c and CH_a meet on the medial line of triangle ABC.

(D. Svhetsov)

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6 The diagonals of convex quadrilateral $ABCD$ are perpendicular. Points A', B', C', D' are the circumcenters of triangles ABD, BCA, CDB, DAC respectively. Prove that lines AA', BB', CC', DD' concur.

(A. Zaslavsky)

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7 Let ABC be an acute-angled, nonisosceles triangle. Altitudes AA' and BB'meet at point H, and the medians of triangle AHB meet at point M. Line CM bisects segment $A'B'$. Find angle C.

(D. Krekov)

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8 A perpendicular bisector of side BC of triangle ABC meets lines AB and AC at points A_B and A_C respectively. Let O_a be the circumcenter of triangle AA_BA_C. Points O_b and O_c are defined similarly. Prove that the circumcircle of triangle $O_aO_bO_c$ touches the circumcircle of the original triangle.

– Grade 10

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1 Let K be an arbitrary point on side BC of triangle ABC, and KN be a bisector of triangle AKC. Lines BN and AK meet at point F, and lines CF and AB meet at point D. Prove that KD is a bisector of triangle AKB.

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2 Prove that an arbitrary triangle with area 1 can be covered by an isosceles triangle with area less than $\sqrt{2}$.

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3 Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB of triangle ABC, respectively. Points B_2 and C_2 are the midpoints of segments BA_1 and CA_1 respectively. Point B_3 is symmetric to C_1 wrt B, and C_3 is symmetric to B_1 wrt C.
Prove that one of common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

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4 Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled, nonisosceles triangle ABC, and A_2, B_2, C_2 be the touching points of sides BC, CA, AB with the correspondent excircles. It is known that line B_1C_1 touches the incircle of ABC.
Prove that A_1 lies on the circumcircle of $A_2B_2C_2$.

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5 Let BM be a median of right-angled nonisosceles triangle ABC ($\angle B = 90$), and H_a, H_c be the orthocenters of triangles ABM, CBM respectively. Lines AH_c and CH_a meet at point K.
Prove that $\angle MBK = 90$.

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6 Let H and O be the orthocenter and the circumcenter of triangle ABC. The circumcircle of triangle AOH meets the perpendicular bisector of BC at point $A_1 \neq O$. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 concur.

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7 Let $SABCD$ be an inscribed pyramid, and AA_1, BB_1, CC_1, DD_1 be the perpendiculars from A, B, C, D to lines SC, SD, SA, SB respectively. Points S, A_1, B_1, C_1, D_1 are distinct and lie on a sphere.
Prove that points A_1, B_1, C_1 and D_1 are coplanar.

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8 Does there exist a rectangle which can be divided into a regular hexagon with sidelength 1 and several congruent right-angled triangles with legs 1 and $\sqrt{3}$?

– Correspondence Round

P1 Tanya cut out a convex polygon from the paper, fold it several times and obtained a two-layers quadrilateral. Can the cutted polygon be a heptagon?

P2 Let O and H be the circumcenter and the orthocenter of a triangle ABC. The line passing through the midpoint of OH and parallel to BC meets AB and AC at points D and E. It is known that O is the incenter of triangle ADE. Find the angles of ABC.

P3 The side AD of a square $ABCD$ is the base of an obtuse-angled isosceles triangle AED with vertex E lying inside the square. Let AF be a diameter of the circumcircle of this triangle, and G be a point on CD such that $CG = DF$. Prove that angle BGE is less than half of angle AED.

P4 In a parallelogram $ABCD$ the trisectors of angles A and B are drawn. Let O be the common points of the trisectors nearest to AB. Let AO meet the second trisector of angle B at point A_1, and let BO meet the second trisector of angle A at point B_1. Let M be the midpoint of A_1B_1. Line MO meets AB at point N Prove that triangle A_1B_1N is equilateral.

P5 Let a triangle ABC be given. Two circles passing through A touch BC at points B and C respectively. Let D be the second common point of these circles (A is closer to BC than D). It is known that $BC = 2BD$. Prove that $\angle DAB = 2\angle ADB$.

P6 Let AA', BB' and CC' be the altitudes of an acute-angled triangle ABC. Points C_a, C_b are symmetric to C' wrt AA' and BB'. Points A_b, A_c, B_c, B_a are defined similarly. Prove that lines A_bB_a, B_cC_b and C_aA_c are parallel.

P7 The altitudes AA_1 and CC_1 of a triangle ABC meet at point H. Point H_A is symmetric to H about A. Line H_AC_1 meets BC at point C', point A' is defined similarly. Prove that $A'C'//AC$.

P8 Diagonals of an isosceles trapezoid $ABCD$ with bases BC and AD are perpendicular. Let DE be the perpendicular from D to AB, and let CF be the perpendicular from C to DE. Prove that angle DBF is equal to half of angle FCD.

P9 Let ABC be an acute-angled triangle. Construct points A', B', C' on its sides BC, CA, AB such that:
- $A'B' \parallel AB$,
- $C'C$ is the bisector of angle $A'C'B'$,
- $A'C' + B'C' = AB$.

P10 The diagonals of a convex quadrilateral divide it into four similar triangles. Prove that is possible to inscribe a circle into this quadrilateral

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P11 Let H be the orthocenter of an acute-angled triangle ABC. The perpendicular bisector to segment BH meets BA and BC at points A_0, C_0 respectively. Prove that the perimeter of triangle A_0OC_0 (O is the circumcenter of triangle ABC) is equal to AC.

P12 Find the maximal number of discs which can be disposed on the plane so that each two of them have a common point and no three have it

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P13 Let AH_1, BH_2 and CH_3 be the altitudes of a triangle ABC. Point M is the midpoint of H_2H_3. Line AM meets H_2H_1 at point K. Prove that K lies on the medial line of ABC parallel to AC.

P14 Let ABC be an acute-angled, nonisosceles triangle. Point A_1, A_2 are symmetric to the feet of the internal and the external bisectors of angle A wrt the midpoint of BC. Segment A_1A_2 is a diameter of a circle α. Circles β and γ are defined similarly. Prove that these three circles have two common points.

P15 The sidelengths of a triangle ABC are not greater than 1. Prove that $p(1 - 2Rr)$ is not greater than 1, where p is the semiperimeter, R and r are the circumradius and the inradius of ABC.

P16 The diagonals of a convex quadrilateral divide it into four triangles. Restore the quadrilateral by the circumcenters of two adjacent triangles and the incenters of two mutually opposite triangles

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P17 Let O be the circumcenter of a triangle ABC. The projections of points D and X to the sidelines of the triangle lie on lines ℓ and L such that $\ell // XO$. Prove that the angles formed by L and by the diagonals of quadrilateral $ABCD$ are equal.

P18 Let $ABCDEF$ be a cyclic hexagon, points K, L, M, N be the common points of lines AB and CD, AC and BD, AF and DE, AE and DF respectively. Prove that if three of these points are collinear then the fourth point lies on the same line.

P19 Let L and K be the feet of the internal and the external bisector of angle A of a triangle ABC. Let P be the common point of the tangents to the circumcircle of the triangle at B and C. The perpendicular from L to BC meets AP at point Q. Prove that Q lies on the medial line of triangle LKP.

P20 Given are a circle and an ellipse lying inside it with focus C. Find the locus of the circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.

P21 A quadrilateral $ABCD$ is inscribed into a circle ω with center O. Let M_1 and M_2 be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OM_1M_2. Let X_1 and

X_2 be the common points of ω and Ω and Y_1 and Y_2 the second common points of Ω with the circumcircles of triangles CDM_1 and ABM_2. Prove that $X_1X_2//Y_1Y_2$.

P22 The faces of an icosahedron are painted into 5 colors in such a way that two faces painted into the same color have no common points, even a vertices. Prove that for any point lying inside the icosahedron the sums of the distances from this point to the red faces and the blue faces are equal.

P23 A tetrahedron $ABCD$ is given. The incircles of triangles ABC and ABD with centers O_1, O_2, touch AB at points T_1, T_2. The plane π_{AB} passing through the midpoint of T_1T_2 is perpendicular to O_1O_2. The planes $\pi_{AC}, \pi_{BC}, \pi_{AD}, \pi_{BD}, \pi_{CD}$ are defined similarly. Prove that these six planes have a common point.

P24 The insphere of a tetrahedron ABCD with center O touches its faces at points A_1, B_1, C_1 and D_1.
a) Let P_a be a point such that its reflections in lines OB, OC and OD lie on plane BCD. Points P_b, P_c and P_d are defined similarly. Prove that lines A_1P_a, B_1P_b, C_1P_c and D_1P_d concur at some point P.
b) Let I be the incenter of $A_1B_1C_1D_1$ and A_2 the common point of line A_1I with plane $B_1C_1D_1$. Points B_2, C_2, D_2 are defined similarly. Prove that P lies inside $A_2B_2C_2D_2$.