Sharygin Geometry Olympiad 2007

www.artofproblemsolving.com/community/c691408

by parmenides51, angemejiabrito, tarzanjunior

–	First (Correspodence) Round
1	A triangle is cut into several (not less than two) triangles. One of them is isosceles (not equilateral), and all others are equilateral. Determine the angles of the original triangle.
2	Each diagonal of a quadrangle divides it into two isosceles triangles. Is it true that the quadrangle is a diamond?
3	Segments connecting an inner point of a convex non-equilateral n-gon to its vertices divide the n-gon into n equal triangles. What is the least possible n?
4	Does a parallelogram exist such that all pairwise meets of bisectors of its angles are situated outside it?
5	A non-convex n-gon is cut into three parts by a straight line, and two parts are put together so that the resulting polygon is equal to the third part. Can n be equal to: a) five? b) four?
6	a) What can be the number of symmetry axes of a checked polygon, that is, of a polygon whose sides lie on lines of a list of checked paper? (Indicate all possible values.) b) What can be the number of symmetry axes of a checked polyhedron, that is, of a polyhedron consisting of equal cubes which border one to another by plane facets?
7	A convex polygon is circumscribed around a circle. Points of contact of its sides with the circle form a polygon with the same set of angles (the order of angles may differ). Is it true that the polygon is regular?
8	Three circles pass through a point P, and the second points of their intersection A, B, C lie on a straight line. Let $A_1 B_1, C_1$ be the second meets of lines AP, BP, CP with the corresponding circles. Let C_2 be the intersections of lines AB_1 and BA_1. Let A_2, B_2 be defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal,
9	Suppose two convex quadrangles are such that the sides of each of them lie on the perpendicular bisectors of the sides of the other one. Determine their angles,

10 Find the locus of centers of regular triangles such that three given points A, B, C lie respectively on three lines containing sides of the triangle.

---

11 A boy and his father are standing on a seashore. If the boy stands on his tiptoes, his eyes are at a height of 1 m above sea-level, and if he seats on father's shoulders, they are at a height of 2 m. What is the ratio of distances visible for him in two eases?
(Find the answer to $0, 1$, assuming that the radius of Earth equals 6000 km.)

---

12 A rectangle $ABCD$ and a point P are given. Lines passing through A and B and perpendicular to PC and PD respectively, meet at a point Q. Prove that $PQ \perp AB$.

---

13 On the side AB of a triangle ABC, two points X, Y are chosen so that $AX = BY$. Lines CX and CY meet the circumcircle of the triangle, for the second time, at points U and V. Prove that all lines UV (for all X, Y, given A, B, C) have a common point.

---

14 In a trapezium with bases AD and BC, let P and Q be the middles of diagonals AC and BD respectively. Prove that if $\angle DAQ = \angle CAB$ then $\angle PBA = \angle DBC$.

---

15 In a triangle ABC, let AA', BB' and CC' be the bisectors. Suppose $A'B' \cap CC' = P$ and $A'C' \cap BB' = Q$. Prove that $\angle PAC = \angle QAB$.

---

16 On two sides of an angle, points A, B are chosen. The midpoint M of the segment AB belongs to two lines such that one of them meets the sides of the angle at points A_1, B_1, and the other at points A_2, B_2. The lines A_1B_2 and A_2B_1 meet AB at points P and Q. Prove that M is the midpoint of PQ.

---

17 What triangles can be cut into three triangles having equal radii of circumcircles?

---

18 Determine the locus of vertices of triangles which have prescribed orthocenter and center of circumcircle.

---

19 Into an angle A of size a, a circle is inscribed tangent to its sides at points B and C. A line tangent to this circle at a point M meets the segments AB and AC at points P and Q respectively. What is the minimum a such that the inequality $S_{PAQ} < S_{BMC}$ is possible?

---

20 The base of a pyramid is a regular triangle having side of size 1. Two of three angles at the vertex of the pyramid are right. Find the maximum value of the volume of the pyramid.

---

21 There are two pipes on the plane (the pipes are circular cylinders of equal size, 4 m around). Two of them are parallel and, being tangent one to another in the common generatrix, form a tunnel over the plane. The third pipe is perpendicular to two others and cuts out a chamber in the tunnel. Determine the area of the surface of this chamber.

– Final Round

– grade 8

1 Determine on which side is the steering wheel disposed in the car depicted in the figure.
https://4.bp.blogspot.com/-s2rjZw-d4UY/XMg5BXCE9SI/AAAAAAAAKHc/WOpvqjWw7lAciDEiNj_TX7io6sfItSPnQCK4BGAYYCw/s320/Sharygin%2Bfinal%2B2007%2B8.1.png

2 By straightedge and compass, reconstruct a right triangle ABC ($\angle C = 90^o$), given the vertices A, C and a point on the bisector of angle B.

3 The diagonals of a convex quadrilateral dissect it into four similar triangles.
Prove that this quadrilateral can also be dissected into two congruent triangles.

4 Determine the locus of orthocenters of triangles, given the midpoint of a side and the feet of the altitudes drawn on two other sides.

5 Medians AA' and BB' of triangle ABC meet at point M, and $\angle AMB = 120^o$.
Prove that angles $AB'M$ and $BA'M$ are neither both acute nor both obtuse.

6 Two non-congruent triangles are called *analogous* if they can be denoted as ABC and $A'B'C'$ such that $AB = A'B', AC = A'C'$ and $\angle B = \angle B'$. Do there exist three mutually *analogous* triangles?

– grade 9

1 Given a circumscribed quadrilateral $ABCD$.
Prove that its inradius is smaller than the sum of the inradii of triangles ABC and ACD.

2 Points E and F are chosen on the base side AD and the lateral side AB of an isosceles trapezoid $ABCD$, respectively. Quadrilateral $CDEF$ is an isosceles trapezoid as well. Prove that $AE \cdot ED = AF \cdot FB$.

3 Given a hexagon $ABCDEF$ such that $AB = BC$, $CD = DE$, $EF = FA$ and $\angle A = \angle C = \angle E$
Prove that AD, BE, CF are concurrent.

4 Given a triangle ABC. An arbitrary point P is chosen on the circumcircle of triangle ABH (H is the orthocenter of triangle ABC). Lines AP and BP meet the opposite sidelines of the triangle at points A' and B', respectively. Determine the locus of midpoints of segments $A'B'$.

5 Reconstruct a triangle, given the incenter, the midpoint of some side and the foot of the altitude drawn on this side.

6 A cube with edge length $2n+1$ is dissected into small cubes of size $1 \times 1 \times 1$ and bars of size $2 \times 2 \times 1$. Find the least possible number of cubes in such a dissection.

– grade 10

1 In an acute triangle ABC, altitudes at vertices A and B and bisector line at angle C intersect the circumcircle again at points A_1, B_1 and C_0. Using the straightedge and compass, reconstruct the triangle by points A_1, B_1 and C_0.

2 Points A', B', C' are the feet of the altitudes AA', BB' and CC' of an acute triangle ABC. A circle with center B and radius BB' meets line $A'C'$ at points K and L (points K and A are on the same side of line BB'). Prove that the intersection point of lines AK and CL belongs to line BO (O is the circumcenter of triangle ABC).

3 Given two circles intersecting at points P and Q. Let C be an arbitrary point distinct from P and Q on the former circle. Let lines CP and CQ intersect again the latter circle at points A and B, respectively. Determine the locus of the circumcenters of triangles ABC.

4 A quadrilateral ABCD is inscribed into a circle with center O. Points C', D' are the reflections of the orthocenters of triangles ABD and ABC at point O. Lines BD and BD' are symmetric with respect to the bisector of angle ABC. Prove that lines AC and AC' are symmetric with respect to the bisector of angle DAB.

5 Each edge of a convex polyhedron is shifted such that the obtained edges form the frame of another convex polyhedron. Are these two polyhedra necessarily congruent?

6 Given are two concentric circles Ω and ω. Each of the circles b_1 and b_2 is externally tangent to ω and internally tangent to Ω, and ω each of the circles c_1 and c_2 is internally tangent to both Ω and ω. Mark each point where one of the circles b_1, b_2 intersects one of the circles c_1 and c_2. Prove that there exist two circles distinct from b_1, b_2, c_1, c_2 which contain all 8 marked points. (Some of these new circles may appear to be lines.)