Olympic Revenge 2015

www.artofproblemsolving.com/community/c691593

by proglote

1 For $n \in \mathbb{N}$, let $P(n)$ denote the product of distinct prime factors of n, with $P(1) = 1$. Show that for any $a_0 \in \mathbb{N}$, if we define a sequence $a_{k+1} = a_k + P(a_k)$ for $k \geq 0$, there exists some $k \in \mathbb{N}$ with $a_k/P(a_k) = 2015$.

---

2 Given $v = (a, b, c, d) \in \mathbb{N}^4$, let $\Delta^1(v) = (|a-b|, |b-c|, |c-d|, |d-a|)$ and $\Delta^k(v) = \Delta(\Delta^{k-1}(v))$ for $k > 1$. Define $f(v) = \min\{k \in \mathbb{N} : \Delta^k(v) = (0,0,0,0)\}$ and $\max(v) = \max\{a, b, c, d\}$. Show that $f(v) < 1000 \log \max(v)$ for all sufficiently large v and $f(v) > 0.001 \log \max(v)$ for infinitely many v.

---

3 For every $n \in \mathbb{N}$, there exist integers k such that $n|k$ and k contains only zeroes and ones in its decimal representation. Let $f(n)$ denote the least possible number of ones in any such k. Determine whether there exists a constant C such that $f(n) < C$ for all $n \in \mathbb{N}$.

---

4 Consider a game in the integer points of the real line, where an Angel tries to escape from a Devil. A positive integer k is chosen, and the Angel and the Devil take turns playing. Initially, no point is blocked. The Angel, in point A, can move to any point P such that $|AP| \leq k$, as long as P is not blocked. The Devil may block an arbitrary point. The Angel loses if it cannot move and wins if it does not lose in finitely many turns. Let $f(k)$ denote the least number of rounds the Devil takes to win. Prove that

$$0.5k\log_2(k)(1+o(1)) \leq f(k) \leq k\log_2(k)(1+o(1)).$$

Note: $a(x) = b(x)(1+o(1))$ if $\lim_{x\to\infty} \frac{b(x)}{a(x)} = 1$.

---

5 Given a triangle $A_1A_2A_3$, let a_i denote the side opposite to A_i, where indices are taken modulo 3. Let $D_1 \in a_1$. For $D_i \in A_i$, let ω_i be the incircle of the triangle formed by lines a_i, a_{i+1}, A_iD_i, and $D_{i+1} \in a_{i+1}$ with $A_{i+1}D_{i+1}$ tangent to ω_i. Show that the set $\{D_i : i \in \mathbb{N}\}$ is finite.
