Spain Mathematical Olympiad 1987

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by parmenides51

– Day 1

1 Let a, b, c be the side lengths of a scalene triangle and let O_a, O_b and O_c be three concentric circles with radii a, b and c respectively.
(a) How many equilateral triangles with different areas can be constructed such that the lines containing the sides are tangent to the circles?
(b) Find the possible areas of such triangles.

2 Show that for each natural number $n > 1$ $1 \cdot \sqrt{\binom{n}{1}} + 2 \cdot \sqrt{\binom{n}{2}} + ... + n \cdot \sqrt{\binom{n}{n}} < \sqrt{2^{n-1} n^3}$

3 A given triangle is divided into n triangles in such a way that any line segment which is a side of a tiling triangle is either a side of another tiling triangle or a side of the given triangle. Let s be the total number of sides and v be the total number of vertices of the tiling triangles (counted without multiplicity).
(a) Show that if n is odd then such divisions are possible, but each of them has the same number v of vertices and the same number s of sides. Express v and s as functions of n.
(b) Show that, for n even, no such tiling is possible

– Day 2

4 If a and b are distinct real numbers, solve the systems
(a) $\begin{cases} x + y = 1 \\ (ax + by)^2 \le a^2 x + b^2 y \end{cases}$ and (b) $\begin{cases} x + y = 1 \\ (ax + by)^4 \le a^4 x + b^4 y \end{cases}$

5 In a triangle ABC, D lies on AB, E lies on AC and $\angle ABE = 30^o, \angle EBC = 50^o, \angle ACD = 20^o$, $\angle DCB = 60^o$. Find $\angle EDC$.

6 For all natural numbers n, consider the polynomial $P_n(x) = x^{n+2} - 2x + 1$.
(a) Show that the equation $P_n(x) = 0$ has exactly one root c_n in the open interval $(0, 1)$.
(b) Find $lim_{n \to \infty} c_n$.