olympiad level geometry problems from Mathematical Reflections

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by parmenides51

O1 A circle centered at O is tangent to all sides of the convex quadrilateral $ABCD$. The rays BA and CD intersect at K, the rays AD and BC intersect at L. The points X, Y are considered on the line segments OA, OC, respectively. Prove that $\angle XKY = \frac{1}{2}\angle AKC$ if and only if $\angle XLY = \frac{1}{2}\angle ALC$.

Proposed by Pavlo Pylyavskyy, MIT

O4 Let AB be a diameter of the circle Γ and let C be a point on the circle, different from A and B. Denote by D the projection of C on AB and let ω be a circle tangent to AD, CD, and Γ, touching Γ at X. Prove that the angle bisectors of $\angle AXB$ and $\angle ACD$ meet on AB.

Proposed by Liubomir Chiriac, Princeton

O7 In the convex hexagon $ABCDEF$ the following equalities hold: $AD = BC + EF, BE = AF + CD, CF = AB + DE$.
Prove that $\frac{AB}{DE} = \frac{CD}{AF} = \frac{EF}{BC}$.

Proposed by Nairi Sedrakyan, Armenia

O13 Let ABC be a triangle and P be an arbitrary point inside the triangle. Let A', B', C', respectively, be the intersections of AP, BP, and CP with the triangles sides. Through P we draw a line perpendicular to PA that intersects BC at A_1. We define B_1 and C_1 analogously. Let P' be the isogonal conjugate of the point P with respect to triangle $A'B'C'$. Show that A_1, B_1, and C_1 lie on a line ℓ that is perpendicular to PP'.

Proposed by Khoa Lu Nguyen, Sam Houston High School, Houston, Texas.

O16 Let ABC be an acute-angled triangle. Let ω be the center of the nine point circle and let G be its centroid. Let $A', B', C', A'', B'', C''$ be the projections of ω and G on the corresponding sides. Prove that the perimeter of $A''B''C''$ is not less than the perimeter of $A'B'C'$.

Proposed by Iurie Boreico, student, Chiinu, Moldova

O20 The incircle of triangle ABC touches AC at E and BC at D. The excircle corresponding to A touches the side BC at A_1 and the extensions of AB, AC at C_1 and B_1, respectively. Let $DE \cap A_1B_1 = L$. Prove that L lies on the circumcircle of triangle A_1BC_1.

Liubomir Chiriac, Princeton University

O22 Consider a triangle ABC and points P, Q in its plane. Let A_1, B_1, C_1 and A_2, B_2, C_2 be cevians in this triangle. Denote by U, V, W the second intersections of circles $(AA_1A_2), (BB_1B_2), (CC_1C_2)$ with circle (ABC), respectively. Let X be the point of intersection of AU with BC. Similarly define Y and Z. Prove that X, Y, Z are collinear.

Khoa Lu Nguyen, M.I.T and Ivan Borsenco, University of Texas at Dallas

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O23 Let ABC be a triangle and let A_1, B_1, C_1 be the points where the angle bisectors of A, B and C meet the circumcircle of triangle ABC, respectively. Let M_a be the midpoint of the segment connecting the intersections of segments A_1B_1 and A_1C_1 with segment BC. Define M_b and M_c analogously. Prove that AM_a, BM_b, and CM_c are concurrent if and only if ABC is isosceles.

Dr. Zuming Feng, Phillips Exeter Academy, New Hampshire

O26 Consider a triangle ABC and let O be its circumcenter. Denote by D the foot of the altitude from A and by E the intersection of AO and BC. Suppose tangents to the circumcircle of triangle ABC at B and C intersect at T and that AT intersects this circumcircle at F. Prove that the circumcircles of triangles DEF and ABC are tangent.

Proposed by Ivan Borsenco, University of Texas at Dallas

O33 Let ABC be a triangle with cicrumcenter O and incenter I. Consider a point M lying on the small arc BC. Prove that $AM + 2OI \geq MB + MC \geq MA - 2OI$

Proposed by Hung Quang Tran, Ha Noi University, Vietnam
