Problems from the iTest Tournament of Champions 2007

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–	Round 1
1	Find the remainder when 3^{2007} is divided by 2007.
2	Let a/b be the probability that a randomly chosen positive divisor of 12^{2007} is also a divisor of 12^{2000}, where a and b are relatively prime positive integers. Find the remainder when $a+b$ is divided by 2007.
3	For each positive integer n, let $g(n)$ be the sum of the digits when n is written in binary. For how many positive integers n, where $1\leq n\leq 2007$, is $g(n)\geq 3$?
4	Black and white coins are placed on some of the squares of a 418×418 grid. All black coins that are in the same row as any white coin(s) are removed. After that, all white coins that are in the same column as any black coin(s) are removed. If b is the number of black coins remaining and w is the number of remaining white coins, find the remainder when the maximum possible value of bw gets divided by 2007.
5	Find the largest possible value of $a+b$ less than or equal to 2007, for which a and b are relatively prime, and such that there is some positive integer n for which $$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1}=\frac{a}{b}.$$
–	Round 2
1	Let a and b be perfect squares whose product exceeds their sum by 4844. Compute the value of $$\left(\sqrt{a}+1\right)\left(\sqrt{b}+1\right)\left(\sqrt{a}-1\right)\left(\sqrt{b}-1\right)-\left(\sqrt{68}+1\right)\left(\sqrt{63}+1\right)\left(\sqrt{68}-1\right)\left(\sqrt{63}-1\right).$$
2	The area of triangle ABC is 2007. One of its sides has length 18, and the tangent of the angle opposite that side is $2007/24832$. When the altitude is dropped to the side of length 18, it cuts that side into two segments. Find the sum of the squares of those two segments.

3 Find the smallest value of n for which the series

$$1 \cdot 3^1 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n$$

exceeds 3^{2007}.

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4 For each positive integer n, let $S_n = \sum_{k=1}^{n} k^3$, and let $d(n)$ be the number of positive divisors of n. For how many positive integers m, where $m \leq 25$, is there a solution n to the equation $d(S_n) = m$?

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5 A polynomial $p(x)$ of degree 1000 is such that $p(n) = (n+1)2^n$ for all nonnegative integers n such that $n \leq 1000$. Given that

$$p(1001) = a \cdot 2^b - c,$$

where a is an odd integer, and $0 < c < 2007$, find $c - (a+b)$.

– Round 3

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1 Let A be the area of the locus of points z in the complex plane that satisfy $|z + 12 + 9i| \leq 15$. Compute $\lfloor A \rfloor$.

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2 Al and Bill play a game involving a fair six-sided die. The die is rolled until either there is a number less than 5 rolled on consecutive tosses, or there is a number greater than 4 on consecutive tosses. Al wins if the last roll is a 5 or 6. Bill wins if the last roll is a 2 or lower. Let m and n be relatively prime positive integers such that m/n is the probability that Bill wins. Find the value of $m + n$.

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3 Find the largest natural number n such that

$$2^n + 2^{11} + 2^8$$

is a perfect square.

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4 Find the smallest positive integer k such that

$$(16a^2 + 36b^2 + 81c^2)(81a^2 + 36b^2 + 16c^2) < k(a^2 + b^2 + c^2)^2,$$

for some ordered triple of positive integers (a, b, c).

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5 Let $s = a + b + c$, where a, b, and c are integers that are lengths of the sides of a box. The volume of the box is numerically equal to the sum of the lengths of the twelve edges of the box plus its surface area. Find the sum of the possible values of s.

– Round 4

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1 A fair 20-sided die has faces numbered 1 through 20. The die is rolled three times and the outcomes are recorded. If a and b are relatively prime integers such that a/b is the probability that the three recorded outcomes can be the sides of a triangle with positive area, find $a+b$.

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2 Let m be the maximum possible value of $x^{16}+\frac{1}{x^{16}}$, where

$$x^6-4x^4-6x^3-4x^2+1=0.$$

Find the remainder when m is divided by 2007.

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3 A sequence $a_1,a_2,a_3,\ldots$ is defined as follows: $a_1=2007$, and $a_n=a_{n-1}+n \pmod{k}$, where $0\le a_n<k$. For how many values of k, where $2007<k<10^{12}$, does the sequence assume all k possible values (modulo k residues)?

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4 Bobby Fisherman played a tournament in which he played 2009 players. He either won or lost every game. He lost his first two games, but won 2002 total games. At the conclusion of each game, he computed his exact winning percentage at that moment. Let $w_1,w_2,\ldots,w_{2009}$ be his winning percentages after games $1,2,\ldots,2009$ respectively. There are some real numbers, such as 0, which are necessarily members of the set $W=\{w_1,w_2,\ldots,w_{2009}\}$. How many positive real numbers are necessarily elements of set W, regardless of the order in which he won or lost his games?

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5 Convex quadrilateral $ABCD$ has the property that the circles with diameters AB and CD are tangent at point X inside the quadrilateral, and likewise, the circles with diameters BC and DA are tangent at a point Y inside the quadrilateral. Given that the perimeter of $ABCD$ is 96, and the maximum possible length of XY is m, find $\lfloor 2007m\rfloor$.

– Round 5

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1 Find the smallest positive integer n such that a cube with sides of length n can be divided up into exactly 2007 smaller cubes, each of whose sides is of integer length.

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2 In the game of *Winners Make Zeros*, a pair of positive integers (m,n) is written on a sheet of paper. Then the game begins, as the players make the following legal moves:

- If $m\ge n$, the player choose a positive integer c such that $m-cn\ge 0$, and replaces (m,n) with $(m-cn,n)$.
- If $m<n$, the player choose a positive integer c such that $n-cm\ge 0$, and replaces (m,n) with $(m,n-cm)$.

When m or n becomes 0, the game ends, and the last player to have moved is declared the winner. If m and n are originally 2007777 and 2007, find the largest choice the first player can make for c (on his first move) such that the first player has a winning strategy after that first move.

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3 Find the sum of all integers n such that

$$n^4 + n^3 + n^2 + n + 1$$

is a perfect square.

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4 In triangle ABC, points A', B', and C' are chosen with A' on segment AB, B' on segment BC, and C' on segment CA so that triangle $A'B'C'$ is directly similar to ABC. The incenters of ABC and $A'B'C'$ are I and I' respectively. Lines BC, $A'C'$, and II' are parallel. If $AB = 100$ and $AC = 120$, what is the length of BC?

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5 Let c be the number of ways to choose three vertices of an 6-dimensional cube that form an equilateral triangle. Find the remainder when c is divided by 2007.

– Round 6

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1 Given that

$$x = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2007},$$
$$y = \frac{1}{1005} + \frac{1}{1006} + \frac{1}{1007} + \cdots + \frac{1}{2007},$$

find the value of k such that

$$x = y + \frac{1}{k}.$$

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2 Let

$$S = 1 + \frac{1}{8} + \frac{1 \cdot 5}{8 \cdot 16} + \frac{1 \cdot 5 \cdot 9}{8 \cdot 16 \cdot 24} + \cdots + \frac{1 \cdot 5 \cdot 9 \cdots (4k+1)}{8 \cdot 16 \cdot 24 \cdots (8k+8)} + \cdots .$$

Find the positive integer n such that $2^n < S^{2007} < 2^{n+1}$.

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3 Find the real number k such that a, b, c, and d are real numbers that satisfy the system of equations

$$abcd = 2007,$$
$$a = \sqrt{55 + \sqrt{k + a}},$$
$$b = \sqrt{55 - \sqrt{k + b}},$$
$$c = \sqrt{55 + \sqrt{k - c}},$$
$$d = \sqrt{55 - \sqrt{k - d}}.$$

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4 Let $x_1, x_2, \ldots, x_{2007}$ be real numbers such that $-1 \le x_i \le 1$ for $1 \le i \le 2007$, and

$$\sum_{i=1}^{2007} x_i^3 = 0.$$

Find the maximum possible value of $\left\lfloor \sum_{i=1}^{2007} x_i \right\rfloor$.

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5 Acute triangle ABC has altitudes AD, BE, and CF. Point D is projected onto AB and AC to points D_c and D_b respectively. Likewise, E is projected to E_a on BC and E_c on AB, and F is projected to F_a on BC and F_b on AC. Lines D_bD_c, E_cE_a, F_aF_b bound a triangle of area T_1, and lines E_cF_b, D_bE_a, F_aD_c bound a triangle of area T_2. What is the smallest possible value of the ratio T_2/T_1?
