China Girls Math Olympiad 2018

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Day 1 August 13, 2018

1 Let $a \leq 1$ be a real number. Sequence $\{x_n\}$ satisfies $x_0 = 0, x_{n+1} = 1 - a \cdot e^{x_n}$, for all $n \geq 1$, where e is the natural logarithm. Prove that for any natural n, $x_n \geq 0$.

2 Points D, E lie on segments AB, AC of $\triangle ABC$ such that $DE \parallel BC$. Let O_1, O_2 be the circumcenters of $\triangle ABE, \triangle ACD$ respectively. Line O_1O_2 meets AC at P, and AB at Q. Let O be the circumcenter of $\triangle APQ$, and M be the intersection of AO extended and BC. Prove that M is the midpoint of BC.

3 Given a real sequence $\{x_n\}_{n=1}^{\infty}$ with $x_1^2 = 1$. Prove that for each integer $n \geq 2$,

$$\sum_{i|n}\sum_{j|n}\frac{x_i x_j}{\text{lcm}\,(i,j)} \geq \prod_{\substack{p \text{ is prime} \\ p|n}}\left(1-\frac{1}{p}\right).$$

4 There're n students whose names are different from each other. Everyone has $n-1$ envelopes initially with the others' name and address written on them respectively. Everyone also has at least one greeting card with her name signed on it. Everyday precisely a student encloses a greeting card (which can be the one received before) with an envelope (the name on the card and the name on envelope cannot be the same) and post it to the appointed student by a same day delivery.
Prove that when no one can post the greeting cards in this way any more:
(i) Everyone still has at least one card;
(ii) If there exist k students $p_1, p_2, \cdots, p_k$ so that p_i never post a card to p_{i+1}, where $i = 1, 2, \cdots, k$ and $p_{k+1} = p_1$, then these k students have prepared the same number of greeting cards initially.

Day 2 August 14, 2018

5 Let $\omega \in \mathbb{C}$, and $|\omega| = 1$. Find the maximum length of $z = (\omega + 2)^3 (\omega - 3)^2$.

6 Given $k \in \mathbb{N}^+$. A sequence of subset of the integer set $\mathbb{Z} \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$ is called a $k-chain$ if for each $1 \leq i \leq k$ we have
(i) $168 \in I_i$;
(ii) $\forall x, y \in I_i$, we have $x - y \in I_i$.
Determine the number of $k-chain$ in total.

7 Given 2018×4 grids and tint them with red and blue. So that each row and each column has the same number of red and blue grids, respectively. Suppose there're M ways to tint the grids with the mentioned requirement. Determine $M \pmod{2018}$.

8 Let I be the incenter of triangle ABC. The tangent point of $\odot I$ on AB, AC is D, E, respectively. Let $BI \cap AC = F, CI \cap AB = G$, $DE \cap BI = M$, $DE \cap CI = N$, $DE \cap FG = P$, $BC \cap IP = Q$. Prove that $BC = 2MN$ is equivalent to $IQ = 2IP$.