Western Mathematical Olympiad 2018

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Day 1 Augest 15th

1 Real numbers $x_1, x_2, \ldots, x_{2018}$ satisfy $x_i + x_j \geq (-1)^{i+j}$ for all $1 \leq i < j \leq 2018$. Find the minimum possible value of $\sum_{i=1}^{2018} i x_i$.

2 Let $n \geq 2$ be an integer. Positive reals $x_1, x_2, \cdots, x_n$ satisfy $x_1 x_2 \cdots x_n = 1$. Show:

$$\{x_1\} + \{x_2\} + \cdots + \{x_n\} < \frac{2n-1}{2}$$

Where $\{x\}$ denotes the fractional part of x.

3 Let $M = \{1, 2, \cdots, 10\}$, and let T be a set of 2-element subsets of M. For any two different elements $\{a, b\}, \{x, y\}$ in T, the integer $(ax+by)(ay+bx)$ is not divisible by 11. Find the maximum size of T.

4 In acute angled $\triangle ABC$, $AB > AC$, points E, F lie on AC, AB respectively, satisfying $BF + CE = BC$. Let I_B, I_C be the excenters of $\triangle ABC$ opposite B, C respectively, EI_C, FI_B intersect at T, and let K be the midpoint of arc BAC. Let KT intersect the circumcircle of $\triangle ABC$ at K, P. Show T, F, P, E concyclic.

Day 2 Augest 16th

5 In acute triangle ABC, $AB < AC$, O is the circumcenter of the triangle. M is the midpoint of segment BC, (AOM) intersects the line AB again at D and intersects the segment AC at E. Prove that $DM = EC$.

6 Let $n \geq 2$ be an integer. Positive reals satisfy $a_1 \geq a_2 \geq \cdots \geq a_n$. Prove that

$$\left(\sum_{i=1}^{n} \frac{a_i}{a_{i+1}}\right) - n \leq \frac{1}{2a_1 a_n} \sum_{i=1}^{n} (a_i - a_{i+1})^2,$$

where $a_{n+1} = a_1$.

7 Let p and c be an prime and a composite, respectively. Prove that there exist two integers m, n, such that

$$0 < m - n < \frac{\text{lcm}(n+1, n+2, \cdots, m)}{\text{lcm}(n, n+1, \cdots, m-1)} = p^c.$$

8 Let n,k be positive integers, satisfying n is even, $k\geq 2$ and $n>4k$. There are n points on the circumference of a circle. If the endpoints of $\frac{n}{2}$ chords in a circle that do not intersect with each other are exactly the n points, we call these chords a matching.Determine the maximum of integer m, such that for any matching, there exists k consecutive points, satisfying all the endpoints of at least m chords are in the k points.