Hong Kong Team Selection Test 2019

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Test 1 August 18, 2018

1 Let a be a real number. Suppose the function $f(x) = \frac{a}{x-1} + \frac{1}{x-2} + \frac{1}{x-6}$ defined in the interval $3 < x < 5$ attains its maximum at $x = 4$. Find the value of a.

2 A circle is circumscribed around an isosceles triangle whose two base angles are equal to x°. Two points are chosen independently and randomly on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is $\frac{14}{25}$. Find the sum of the largest and smallest possible value of x.

3 Find an integral solution of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \dots + \left\lfloor \frac{x}{10!} \right\rfloor = 2019.$$

(Note $\lfloor u \rfloor$ stands for the greatest integer less than or equal to u.)

4 Let ABC be an acute-angled triangle such that $\angle ACB = 45^\circ$. Let G be the point of intersection of the three medians and let O be the circumcentre. Suppose $OG = 1$ and $OG \parallel BC$. Determine the length of the segment BC.

5 Is it is possible to choose 24 distinct points in the space such that no three of them lie on the same line and choose 2019 distinct planes in a way that each plane passes through at least 3 of the chosen points and each triple belongs to one of the chosen planes?

6 If $57a + 88b + 125c \geq 1148$, where $a, b, c > 0$, what is the minimum value of

$$a^3 + b^3 + c^3 + 5a^2 + 5b^2 + 5c^2?$$

Test 2 October 20, 2018

1 Determine all sequences $p_1, p_2, \dots$ of prime numbers for which there exists an integer k such that the recurrence relation

$$p_{n+2} = p_{n+1} + p_n + k$$

holds for all positive integers n.

2 Let p be a prime number greater than 10. Prove that there exist positive integers m and n such that $m+n<p$ and 5^m7^n-1 is divisible by p.

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3 Let Γ_1 and Γ_2 be two circles with different radii, with Γ_1 the smaller one. The two circles meet at distinct points A and B. C and D are two points on the circles Γ_1 and Γ_2, respectively, and such that A is the midpoint of CD. CB is extended to meet Γ_2 at F, while DB is extended to meet Γ_1 at E. The perpendicular bisector of CD and the perpendicular bisector of EF meet at P.

(a) Prove that $\angle EPF = 2\angle CAE$.
(b) Prove that $AP^2 = CA^2 + PE^2$.

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4 We choose 100 points in the coordinate plane. Let N be the number of triples (A,B,C) of distinct chosen points such that A and B have the same y-coordinate, and B and C have the same x-coordinate. Find the greatest value that N can attain considering all possible ways to choose the points.

Note The CHKMO and APMO are used as selection tests in between these Test 2 and 3.

Test 3 April 28, 2019

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1 Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f:\mathbb{Q}_{>0}\to\mathbb{Q}_{>0}$ satisfying

$$f(x^2f(y)^2)=f(x)^2f(y)$$

for all $x,y\in\mathbb{Q}_{>0}$

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2 Let ABC be a triangle with $AB=AC$, and let M be the midpoint of BC. Let P be a point such that $PB<PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

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3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil\frac{n}{1}\right\rceil+\left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{n}{3}\right\rceil+\cdots+\left\lceil\frac{n}{n}\right\rceil$$

turns. (As usual, $\lceil x\rceil$ stands for the least integer not smaller than x.)

Test 4 May 1, 2019

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1 Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

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2 Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

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3 Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

Proposed by Evan Chen, Taiwan
