Vietnam National Olympiad 1972

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1 Let α be an arbitrary angle and let $x = cos\alpha, y = cosn\alpha$ ($n \in Z$).
i) Prove that to each value $x \in [-1, 1]$ corresponds one and only one value of y.
Thus we can write y as a function of $x, y = T_n(x)$.
Compute $T_1(x), T_2(x)$ and prove that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.
From this it follows that $T_n(x)$ is a polynomial of degree n.
ii) Prove that the polynomial $T_n(x)$ has n distinct roots in $[-1, 1]$.

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3 ABC is a triangle. U is a point on the line BC. I is the midpoint of BC. The line through C parallel to AI meets the line AU at E. The line through E parallel to BC meets the line AB at F. The line through E parallel to AB meets the line BC at H. The line through H parallel to AU meets the line AB at K. The lines HK and FG meet at $T.V$ is the point on the line AU such that A is the midpoint of UV. Show that V, T and I are collinear.

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4 Let $ABCD$ be a regular tetrahedron with side a. Take E, E' on the edge AB, F, F' on the edge AC and G, G' on the edge AD so that $AE = a/6, AE' = 5a/6, AF = a/4, AF' = 3a/4, AG = a/3, AG' = 2a/3$. Compute the volume of $EFGE'F'G'$ in term of a and find the angles between the lines AB, AC, AD and the plane EFG.
