Czech and Slovak Match 1996

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–	Day 1
1	Show that an integer $p > 3$ is a prime if and only if for every two nonzero integers a, b exactly one of the numbers $N_1 = a + b - 6ab + \frac{p-1}{6}$, $N_2 = a + b + 6ab + \frac{p-1}{6}$ is a nonzero integer.
2	Let be a binary operation on a nonempty set M. That is, every pair $(a, b) \in M$ is assigned an element $a\ b$ in M. Suppose that has the additional property that $(a\ \ b)\ b = a$ and $a\ \ (a\ b) = b$ for all $a, b \in M$. (a) Show that $a\ \ b = b\ \ a$ for all $a, b \in M$. (b) On which finite sets M does such a binary operation exist?
3	The base of a regular quadrilateral pyramid π is a square with side length $2a$ and its lateral edge has length a$\sqrt{17}$. Let M be a point inside the pyramid. Consider the five pyramids which are similar to π , whose top vertex is at M and whose bases lie in the planes of the faces of π . Show that the sum of the surface areas of these five pyramids is greater or equal to one fifth the surface of π , and find for which M equality holds.
–	Day 2
4	Decide whether there exists a function $f : Z \to Z$ such that for each $k = 0, 1, ..., 1996$ and for any integer m the equation $f(x) + kx = m$ has at least one integral solution x.
5	Two sets of intervals A, B on the line are given. The set A contains $2m - 1$ intervals, every two of which have an interior point in common. Moreover, every interval from A contains at least two disjoint intervals from B. Show that there exists an interval in B which belongs to at least m intervals from A .
6	The points E and D lie in the interior of sides AC and BC, respectively, of a triangle ABC. Let F be the intersection of the lines AD and BE.Show that the area of the traingles ABC and ABF satisfies: $\frac{S_{ABC}}{S_{ABF}} = \frac{\|AC\|}{\|AE\|} + \frac{\|BC\|}{\|BD\|} - 1.$