Czech and Slovak Match 2000

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–	Day 1
1	a, b, c are positive real numbers which satisfy $5abc > a^3 + b^3 + c^3$. Prove that a, b, c can form a triangle.
2	Let ABC be a triangle, k its incircle and k_a, k_b, k_c three circles orthogonal to k passing through B and C, A and C , and A and B respectively. The circles k_a, k_b meet again in C' ; in the same way we obtain the points B' and A' . Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of k .
3	Let n be a positive integer. Prove that n is a power of two if and only if there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.
–	Day 2
4	Let $P(x)$ be a polynomial with integer coefficients. Prove that the polynomial $Q(x) = P(x^4)P(x^3)P(x^2)P(x)$ 1 has no integer roots.
5	Let $ABCD$ be an isosceles trapezoid with bases AB and CD. The incircle of the triangle BCD touches CD at E. Point F is chosen on the bisector of the angle DAC such that the lines EF and CD are perpendicular. The circumcircle of the triangle ACF intersects the line CD again at G. Prove that the triangle AFG is isosceles.
6	Suppose that every integer has been given one of the colors red, blue, green, yellow. Let x and y be odd integers such that $\|x\| \neq \|y\|$. Show that there are two integers of the same color whose difference has one of the following values: $x, y, x + y, x - y$.