Singapore Team Selection Test 2009

www.artofproblemsolving.com/community/c708172

by parmenides51, Agung, Tales, dominicleejun, April

– Day 1

1 Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that the line TP bisects $\angle ATB$.

2 If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\leq\frac{1}{abc}.$$

3 Determine the smallest positive integer N such that there exists 6 distinct integers $a_1, a_2, a_3, a_4, a_5, a_6 > 0$ satisfying:
(i) $N = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
(ii) $N - a_i$ is a perfect square for $i = 1, 2, 3, 4, 5, 6$.

– Day 2

1 Let $S = \{a + np : n = 0, 1, 2, 3, ...\}$ where a is a positive integer and p is a prime. Suppose there exist positive integers x and y st x^{41} and y^{49} are in S. Determine if there exists a positive integer z st z^{2009} is in S.

2 Let H be the orthocentre of $\triangle ABC$ and let P be a point on the circumcircle of $\triangle ABC$, distinct from A, B, C. Let E and F be the feet of altitudes from H onto AC and AB respectively. Let $PAQB$ and $PARC$ be parallelograms. Suppose QA meets RH at X and RA meets QH at Y. Prove that XE is parallel to YF.

3 In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes B_1, . . ., B_n such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod n$.

Proposed by Gerhard Woeginger, Netherlands