Vietnam Team Selection Test 2018

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– Day 1

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1 Let ABC be a acute, non-isosceles triangle. $D,\ E,\ F$ are the midpoints of sides $AB,\ BC,\ AC$, resp. Denote by $(O),\ (O')$ the circumcircle and Euler circle of ABC. An arbitrary point P lies inside triangle DEF and $DP,\ EP,\ FP$ intersect (O') at $D',\ E',\ F'$, resp. Point A' is the point such that D' is the midpoint of AA'. Points $B',\ C'$ are defined similarly.
a. Prove that if $PO = PO'$ then $O \in (A'B'C')$;
b. Point A' is mirrored by OD, its image is X. $Y,\ Z$ are created in the same manner. H is the orthocenter of ABC and $XH,\ YH,\ ZH$ intersect BC, AC, AB at $M,\ N,\ L$ resp. Prove that $M,\ N,\ L$ are collinear.

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2 For every positive integer m, a $m \times 2018$ rectangle consists of unit squares (called "cell") is called *complete* if the following conditions are met:
i. In each cell is written either a "0", a "1" or nothing;
ii. For any binary string S with length 2018, one may choose a row and complete the empty cells so that the numbers in that row, if read from left to right, produce S (In particular, if a row is already full and it produces S in the same manner then this condition ii. is satisfied).
A *complete* rectangle is called *minimal*, if we remove any of its rows and then making it no longer *complete*.

a. Prove that for any positive integer $k \leq 2018$ there exists a *minimal* $2^k \times 2018$ rectangle with exactly k columns containing both 0 and 1.
b. A *minimal* $m \times 2018$ rectangle has exactly k columns containing at least some 0 or 1 and the rest of columns are empty. Prove that $m \leq 2^k$.

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3 For every positive integer $n \geq 3$, let ϕ_n be the set of all positive integers less than and coprime to n. Consider the polynomial:

$$P_n(x) = \sum_{k \in \phi_n} x^{k-1}.$$

a. Prove that $P_n(x) = (x^{r_n} + 1)Q_n(x)$ for some positive integer r_n and polynomial $Q_n(x) \in \mathbb{Z}[x]$ (not necessary non-constant polynomial).
b. Find all n such that $P_n(x)$ is irreducible over $\mathbb{Z}[x]$.

– Day 2

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4 Let $a \in \left[\frac{1}{2},\ \frac{3}{2}\right]$ be a real number. Sequences $(u_n),\ (v_n)$ are defined as follows:

$$u_n = \frac{3}{2^{n+1}} \cdot (-1)^{\lfloor 2^{n+1}a \rfloor},\ v_n = \frac{3}{2^{n+1}} \cdot (-1)^{n+\lfloor 2^{n+1}a \rfloor}.$$

a. Prove that

$$(u_0 + u_1 + \cdots + u_{2018})^2 + (v_0 + v_1 + \cdots + v_{2018})^2 \le 72a^2 - 48a + 10 + \frac{2}{4^{2019}}.$$

b. Find all values of a in the equality case.

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5 In a $m \times n$ square grid, with top-left corner is A, there is route along the edges of the grid starting from A and visits all lattice points (called "nodes") exactly once and ending also at A.

a. Prove that this route exists if and only if at least one of $m,\ n$ is odd.
b. If such a route exists, then what is the least possible of turning points?

*A turning point is a node that is different from A and if two edges on the route intersect at the node are perpendicular.

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6 Triangle ABC circumscribed (O) has A-excircle (J) that touches $AB,\ BC,\ AC$ at $F,\ D,\ E$, resp.

a. L is the midpoint of BC. Circle with diameter LJ cuts $DE,\ DF$ at $K,\ H$. Prove that $(BDK),\ (CDH)$ has an intersecting point on (J).
b. Let $EF \cap BC = \{G\}$ and GJ cuts $AB,\ AC$ at $M,\ N$, resp. $P \in JB$ and $Q \in JC$ such that

$$\angle PAB = \angle QAC = 90^\circ.$$

$PM \cap QN = \{T\}$ and S is the midpoint of the larger BC-arc of (O). (I) is the incircle of ABC. Prove that $SI \cap AT \in (O)$.
