Vietnam Team Selection Test 2013

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– Day 1

1 The $ABCD$ is a cyclic quadrilateral with no parallel sides inscribed in circle (O,R). Let E be the intersection of two diagonals and the angle bisector of AEB cut the lines AB,BC,CD,DA at M,N,P,Q respectively .
a) Prove that the circles $(AQM),(BMN),(CNP),(DPQ)$ are passing through a point. Call that point K.
b) Denote $min\left \{ AC,BD \right \}=m$. Prove that $OK\leq \dfrac{2R^2}{\sqrt{4R^2-m^2}}$.

2 a. Prove that there are infinitely many positive integers t such that both $2012t+1$ and $2013t+1$ are perfect squares.
b. Suppose that m,n are positive integers such that both $mn+1$ and $mn+n+1$ are perfect squares. Prove that $8(2m+1)$ divides n.

3 Given a number $n\in\mathbb{Z}^+$ and let S denotes the set $\{0,1,2,...,2n+1\}$. Consider the function $f:\mathbb{Z}\times S\to [0,1]$ satisfying two following conditions simultaneously:

i) $f(x,0)=f(x,2n+1)=0\forall x\in\mathbb{Z}$;
ii) $f(x-1,y)+f(x+1,y)+f(x,y-1)+f(x,y+1)=1$ for all $x\in\mathbb{Z}$ and $y\in\{1,2,3,...,2n\}$.

Let F be the set of such functions. For each $f\in F$, let $v(f)$ be the set of values of f.

a) Proof that $|F|=\infty$.

b) Proof that for each $f\in F$ then $|v(f)|<\infty$.

c) Find the maximum value of $|v(f)|$ for $f\in F$.

– Day 2

4 Find the greatest positive integer k such that the following inequality holds for all $a,b,c\in\mathbb{R}^+$ satisfying $abc=1$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{k}{a+b+c+1}\geqslant 3+\frac{k}{4}$$

5 Let ABC be a triangle with $\angle BAC=45^o$. Altitudes AD,BE,CF meet at H. EF cuts BC at P. I is the midpoint of BC, IF cuts PH in Q.
a) Prove that $\angle IQH=\angle AIE$.

b) Let (K) be the circumcircle of triangle ABC, (J) be the circumcircle of triangle KPD. CK cuts circle (J) at G, IG cuts (J) at M, JC cuts circle of diameter BC at N. Prove that G, N, M, C lie on the same circle.

6 A cube with size $10 \times 10 \times 10$ consists of 1000 unit cubes, all colored white. A and B play a game on this cube. A chooses some pillars with size $1 \times 10 \times 10$ such that no two pillars share a vertex or side, and turns all chosen unit cubes to black. B is allowed to choose some unit cubes and ask A their colors. How many unit cubes, at least, that B need to choose so that for any answer from A, B can always determine the black unit cubes?