Vietnam Team Selection Test 2014

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– Day 1

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1 Find all $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(2m+f(m)+f(m)f(n))=nf(m)+m$$

$\forall m,n\in\mathbb{Z}$

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2 In the Cartesian plane is given a set of points with integer coordinate

$$T=\{(x;y)\mid x,y\in\mathbb{Z};\ |x|,|y|\leq 20;\ (x;y)\neq(0;0)\}$$

We colour some points of T such that for each point $(x;y)\in T$ then either $(x;y)$ or $(-x;-y)$ is coloured. Denote N to be the number of couples $(x_1;y_1),(x_2;y_2)$ such that both $(x_1;y_1)$ and $(x_2;y_2)$ are coloured and $x_1\equiv 2x_2\pmod{41}, y_1\equiv 2y_2\pmod{41}$. Find the all possible values of N.

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3 Let ABC be triangle with $A<B<C$ and inscribed in a circle (O). On the minor arc ABC of (O) and does not contain point A, choose an arbitrary point D. Suppose CD meets AB at E and BD meets AC at F. Let O_1 be the incenter of triangle EBD touches with EB,ED and tangent to (O). Let O_2 be the incenter of triangle FCD, touches with FC,FD and tangent to (O).
a) M is a tangency point of O_1 with BE and N is a tangency point of O_2 with CF. Prove that the circle with diameter MN has a fixed point.
b) A line through M is parallel to CE meets AC at P, a line through N is parallel to BF meets AB at Q. Prove that the circumcircles of triangles $(AMP),(ANQ)$ are all tangent to a fixed circle.

– Day 2

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4 a. Let ABC be a triangle with altitude AD and P a variable point on AD. Lines PB and AC intersect each other at E, lines PC and AB intersect each other at F. Suppose $AEDF$ is a quadrilateral inscribed . Prove that

$$\frac{PA}{PD}=(\tan B+\tan C)\cot\frac{A}{2}.$$

b. Let ABC be a triangle with orthocentre H and P a variable point on AH. The line through C perpendicular to AC meets BP at M, The line through B perpendicular to AB meets CP at N. K is the projection of Aon MN. Prove that $\angle BKC+\angle MAN$ is invariant .

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5 Find all polynomials $P(x), Q(x)$ which have integer coefficients and satify the following condtion: For the sequence (x_n) defined by

$$x_0 = 2014, x_{2n+1} = P(x_{2n}), x_{2n} = Q(x_{2n-1}) \quad n \geq 1$$

for every positive integer m is a divisor of some non-zero element of (x_n)

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6 m, n, p are positive integers which do not simultaneously equal to zero. 3D Cartesian space is divided into unit cubes by planes each perpendicular to one of 3 axes and cutting corresponding axis at integer coordinates. Each unit cube is filled with an integer from 1 to 60. A filling of integers is called *Dien Bien* if, for each rectangular box of size $\{2m+1, 2n+1, 2p+1\}$, the number in the unit cube which has common centre with the rectangular box is the average of the 8 numbers of the 8 unit cubes at the 8 corners of that rectangular box. How many *Dien Bien* fillings are there?
Two fillings are the same if one filling can be transformed to the other filling via a translation.

translation from here (`http://artofproblemsolving.com/community/c6h592875p3515526`)
