Rioplatense Mathematical Olympiad, Level 3 2011

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–	Day 1
1	Given a positive integer n, an operation consists of replacing n with either $2n-1$, $3n-2$ or $5n-4$. A number b is said to be a *follower* of number a if b can be obtained from a using this operation multiple times. Find all positive integers $a < 2011$ that have a common follower with 2011.
2	Let ABC an acute triangle and H its orthocenter. Let E and F be the intersection of lines BH and CH with AC and AB respectively, and let D be the intersection of lines EF and BC. Let Γ_1 be the circumcircle of AEF, and Γ_2 the circumcircle of BHC. The line AD intersects Γ_1 at point $I \neq A$. Let J be the feet of the internal bisector of $\angle BHC$ and M the midpoint of the arc $\overset{\frown}{BC}$ from Γ_2 that contains the point H. The line MJ intersects Γ_2 at point $N \neq M$. Show that the triangles EIF and CNB are similar.
3	Let M be a map made of several cities linked to each other by flights. We say that there is a route between two cities if there is a nonstop flight linking these two cities. For each city to the M denote by M_a the map formed by the cities that have a route to and routes linking these cities together (to not part of M_a). The cities of M_a are divided into two sets so that the number of routes linking cities of different sets is maximum; we call this number the cut of M_a. Suppose that for every cut of M_a , it is strictly less than two thirds of the number of routes M_a . Show that for any coloring of the M routes with two colors there are three cities of M joined by three routes of the same color.
–	Day 2
4	We consider Γ_1 and Γ_2 two circles that intersect at points P and Q . Let A, B and C be points on the circle Γ_1 and D, E and F points on the circle Γ_2 so that the lines AE and BD intersect at P and the lines AF and CD intersect at Q. Denote M and N the intersections of lines AB and DE and of lines AC and DF , respectively. Show that $AMDN$ is a parallelogram.
5	A *form* is the union of squared rectangles whose bases are consecutive unitary segments in a horizontal line that leaves all the rectangles on the same side, and whose heights $m_1, ..., m_n$ satisying $m_1 \geq ... \geq m_n$. An *angle* in a *form* consists of a box v and of all the boxes to the right of v and all the boxes above v. The size of a *form* of an *angle* is the number of boxes it contains. Find the maximum number of *angles* of size 11 in a form of size 400. source (`http://www.oma.org.ar/enunciados/omr20.htm`)

6 Let $d(n)$ be the sum of positive integers divisors of number n and $\phi(n)$ the quantity of integers in the interval $[0, n]$ such that these integers are coprime with n. For instance $d(6) = 12$ and $\phi(7) = 6$.
Determine if the set of the integers n such that, $d(n) \cdot \phi(n)$ is a perfect square, is finite or infinite set.