Rioplatense Mathematical Olympiad, Level 3 2014

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–	Day 1
1	Let $n \geq 3$ be a positive integer. Determine, in terms of n, how many triples of sets (A, B, C) satisfy the conditions: • A, B and C are pairwise disjoint , that is, $A \cap B = A \cap C = B \cap C = \emptyset$. • $A \cup B \cup C = \{1, 2, ..., n\}$. • The sum of the elements of A, the sum of the elements of B and the sum of the elements of C leave the same remainder when divided by 3. Note: One or more of the sets may be empty.
2	El Chapulín observed that the number 2014 has an unusual property. By placing its eight positive divisors in increasing order, the fifth divisor is equal to three times the third minus 4. A number of eight divisors with this unusual property is called the *red* number . How many *red* numbers smaller than 2014 exist?
3	Kiko and Ñoño play with a rod of length $2n$ where $n \leq 3$ is an integer. Kiko cuts the rod in $k \leq 2n$ pieces of integer lengths. Then Ñoño has to arrange these pieces so that they form a hexagon of equal opposite sides and equal angles. The pieces can not be split and they all have to be used. If Ñoño achieves his goal, he wins, in any other case, Kiko wins. Determine which victory can be secured based on k.
–	Day 2
4	A pair (a,b) of positive integers is *Rioplatense* if it is true that $b + k$ is a multiple of $a + k$ for all $k \in \{0, 1, 2, 3, 4\}$. Prove that there is an infinite set A of positive integers such that for any two elements a and b of A, with $a < b$, the pair (a, b) is *Rioplatense*.
5	In the segment AC a point B is taken. Construct circles T_1, T_2 and T_3 of diameters AB, BC and AC respectively. A line that passes through B cuts T_3 in the points P and Q, and the circles T_1 and T_2 respectively at points R and S. Prove that $PR = QS$.
6	Let $n \in N$ such that $1 + 2 + ... + n$ is divisible by 3. Integers $a_1 \geq a_2 \geq a_3 \geq 2$ have sum n and they satisfy $1 + 2 + ... + a_1 \leq \frac{1}{3}(1 + 2 + ... + n)$ and $1 + 2 + ... + (a_1 + a_2) \leq \frac{2}{3}(1 + 2 + ... + n)$. Prove that there is a partition of $\{1, 2, ..., n\}$ in three subsets A_1, A_2, A_3 with cardinals $\|A_i\| = a_i, i = 1, 2, 3$, and with equal sums of their elements .