Rioplatense Mathematical Olympiad, Level 3 2016

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by parmenides51, mathisreal

– Day 1

1 Ana and Beto play against each other. Initially, Ana chooses a non-negative integer N and announces it to Beto. Next Beto writes a succession of 2016 numbers, 1008 of them equal to 1 and 1008 of them equal to -1. Once this is done, Ana must split the succession into several blocks of consecutive terms (each term belonging to exactly one block), and calculate the sum of the numbers of each block. Finally, add the squares of the calculated numbers. If this sum is equal to N, Ana wins. If not, Beto wins. Determine all values of N for which Ana can ensure victory, no matter how Beto plays.

2 Determine all positive integers n for which there are positive real numbers x, y and z such that $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and $\sqrt{x+n}+\sqrt{y+n}+\sqrt{z+n}$ is an integer.

3 Let ABC be an acute-angled triangle of circumcenter O and orthocenter H. Let M be the midpoint of BC, N be the symmetric of H with respect to A, P be the midpoint of NM and X be a point on the line A H such that MX is parallel to CH. Prove that BX and OP are perpendicular.

– Day 2

4 Let $c > 1$ be a real number. A function $f : [0, 1] \to R$ is called c-friendly if $f(0) = 0, f(1) = 1$ and $|f(x) - f(y)| \le c|x - y|$ for all the numbers $x, y \in [0, 1]$. Find the maximum of the expression $|f(x) - f(y)|$ for all *c-friendly* functions f and for all the numbers $x, y \in [0, 1]$.

5 Initially one have the number 0 in each cell of the table 29×29. A *moviment* is when one choose a sub-table 5×5 and add $+1$ for every cell of this sub-table. Find the maximum value of n, where after 1000 *moviments*, there are 4 cells such that your centers are vertices of a square and the sum of this 4 cells is at least n.
Note: A square does not, necessarily, have your sides parallel with the sides of the table.

6 When the natural numbers are written one after another in an increasing way, you get an infinite succession of digits $123456789101112....$ Denote A_k the number formed by the first k digits of this sequence . Prove that for all positive integer n there is a positive integer m which simultaneously verifies the following three conditions:
(i) n divides A_m,
(ii) n divides m,
(iii) n divides the sum of the digits of A_m.

