National Math Olympiad (3rd Round) 2018

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–	Combinatorics
1	Alice and Bob are play a game in a $(2n)*(2n)$ chess boared.Alice starts from the top right point moving 1 unit in every turn.Bob starts from the down left square and does the same.The goal of Alice is to make a closed shape ending in its start position and cannot reach any point that was reached before by any of players .if a players cannot move the other player keeps moving.what is the maximum are of the shape that the first player can build with every strategy of second player.
2	There are 8 points in the plane.we write down the area of each triangle having all vertices amoung these points(totally 56 numbers).Let them be $a_1, a_2, \dots a_{56}$.Prove that there is a choice of plus or minus such that: $$\pm a_1 \pm a_2 \dots \pm a_{56} = 0$$
3	Find the smallest positive integer n such that we can write numbers $1, 2, \dots, n$ in a 18*18 board such that: i)each number appears at least once ii)In each row or column,there are no two numbers having difference 0 or 1
4	Let n be a positive integer.Consider all 2^n binary strings of length n.We say two of these strings are neighbors if they differ in exactly 1 digit.We have colored m strings.In each moment,we can color any uncolored string which is neighbor with at least 2 colored strings.After some time,all the strings are colored.Find the minimum possible value of m.
–	Geometry
1	Incircle of triangle ABC is tangent to sides BC, CA, AB at D, E, F,respectively.Points P, Q are inside angle BAC such that $FP = FB, FP \| \| AC$ and $EQ = EC, EQ \| \| AB$.Prove that P, Q, D are collinear.
2	Two intersecting circles ω_1 and ω_2 are given.Lines AB, CD are common tangents of $\omega_1, \omega_2 (A, C \in \omega_1, B, D \in \omega_2)$ Let M be the midpoint of AB.Tangents through M to ω_1 and ω_2(other than AB) intersect CD at X, Y.Let I be the incenter of MXY.Prove that $IC = ID$.

3 H is the orthocenter of acude triangle ABC.Let ω be the circumcircle of BHC with center O'.Ω is the nine-point circle of ABC.X is an arbitrary point on arc BHC of ω and AX intersects Ω at Y.P is a point on Ω such that $PX = PY$.Prove that $O'PX = 90$.

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4 for acute triangle $\triangle ABC$ with orthocenter H, and E, F the feet of altitudes for B, C, we have P on EF such as that $HO \perp HP$. Q is on segment AH so $HM \perp PQ$. prove $QA = 3QH$

– Number theory

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1 $n \geq 2$ is an integer.Prove that the number of natural numbers m so that $0 \leq m \leq n^2 - 1, x^n + y^n \equiv m(modn^2)$ has no solutions is at least $\binom{n}{2}$

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2 Prove that for every prime number p there exist infinity many natural numbers n so that they satisfy:

$2^{2^{2^{\dots^{2^n}}}} \equiv n^{2^{2^{\dots^{2}}}}(modp)$

Where in both sides 2 appeared 1397 times

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3 Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that for every natural numbers m, n :$f(n) + 2mn + f(m)$ is a perfect square.

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4 Prove that for any natural numbersa, b there exist infinity many prime numbers p so that $Ord_p(a) = Ord_p(b)$(Proving that there exist infinity prime numbers p so that $Ord_p(a) \geq Ord_p(b)$ will get a partial mark)

– Algebra

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1 For positive real numbersa, b, csuch that $ab + ac + bc = 1$ prove that:

$\prod_{cyc}(\sqrt{bc} + \frac{1}{2a+\sqrt{bc}}) \geq 8abc$

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2 Find all functions $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that:

$f(x^3 + xf(xy)) = f(xy) + x^2 f(x + y) \forall x, y \in \mathbb{R}^{\geq 0}$

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3 A)Let x, y be two complex numbers on the unit circle so that:

$\frac{\pi}{3} \leq \arg(x) - \arg(y) \leq \frac{5\pi}{3}$

Prove that for any $z \in \mathbb{C}$ we have:

$|z| + |z - x| + |z - y| \geq |zx - y|$

B)Let x, y be two complex numbers so that:

$\frac{\pi}{3} \le \arg(x) - \arg(y) \le \frac{2\pi}{3}$

Prove that for any $z \in \mathbb{C}$ we have:

$|z| + |z-y| + |z-x| \ge |\frac{\sqrt{3}}{2}x + (y - \frac{x}{2})i|$

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4 Let $P(x)$ be a non-zero polynomial with real coefficient so that $P(0) = 0$.Prove that for any positive real number M there exist a positive integer d so that for any monic polynomial $Q(x)$ with degree at least d the number of integers k so that $|P(Q(k))| \le M$ is at most equal to the degree of Q.
