Silk Road Mathematics Competiton 2006

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1 Found all functions $f:\mathbb{R}\to\mathbb{R}$, such that for any $x,y\in\mathbb{R}$,

$$f(x^2+xy+f(y))=f^2(x)+xf(y)+y.$$

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2 For positive a,b,c, such that $abc=1$ prove the inequality: $4(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{c}}+\sqrt[3]{\frac{c}{a}})\leq 3(2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})^{\frac{2}{3}}$.

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3 A subset S of the set $M=\{1,2,.....,p-1\}$,where p is a prime number of the kind $12n+11$,is *essential*,if the product Π_s of all elements of the subset is not less than the product $\bar{\Pi_s}$ of all other elements of the set.The **difference** $\bigtriangleup_s=\Pi_s-\bar{\Pi_s}$ is called *the deviation* of the subset S.Define the least possible remainder of division by p of the deviation of an essential subset,containing $\frac{p-1}{2}$ elements.

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4 A family L of 2006 lines on the plane is given in such a way that it doesn't contain parallel lines and it doesn't contain three lines with a common point.We say that the line $l_1\in L$ is *bounding* the line $l_2\in L$,if all intersection points of the line l_2 with other lines from L lie on the one side of the line l_1.
Prove that in the family L there are two lines l and l' such that the following 2 conditions are satisfied simultaneously:
1) The line l is bounding the line l';
2) the line l' is not bounding the line l.
