Silk Road Mathematics Competiton 2015

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by MRF2017, parmenides51, rightways

1 Prove that there is no positive real numbers a, b, c, d such that both of the following equations hold.

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=6, \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}=32$$

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2 Let $\{a_n\}_{n\ge 1}$ and $\{b_n\}_{n\ge 1}$ be two infinite arithmetic progressions, each of which the first term and the difference are mutually prime natural numbers. It is known that for any natural n, at least one of the numbers $\left(a_n^2+a_{n+1}^2\right)\left(b_n^2+b_{n+1}^2\right)$ or $\left(a_n^2+b_n^2\right)\left(a_{n+1}^2+b_{n+1}^2\right)$ is an perfect square. Prove that $a_n=b_n$, for any natural n .

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3 Let B_n be the set of all sequences of length n, consisting of zeros and ones. For every two sequences $a, b \in B_n$ (not necessarily different) we define strings $\varepsilon_0\varepsilon_1\varepsilon_2\dots\varepsilon_n$ and $\delta_0\delta_1\delta_2\dots\delta_n$ such that $\varepsilon_0=\delta_0=0$ and

$$\varepsilon_{i+1}=(\delta_i-a_{i+1})(\delta_i-b_{i+1}), \quad \delta_{i+1}=\delta_i+(-1)^{\delta_i}\varepsilon_{i+1} \quad (0\le i\le n-1).$$

. Let $w(a,b)=\varepsilon_0+\varepsilon_1+\varepsilon_2+\dots+\varepsilon_n$. Find $f(n)=\sum\limits_{a,b\in B_n} w(a,b)$.

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4 Let O be a circumcenter of an acute-angled triangle ABC. Consider two circles ω and Ω inscribed in the angle $\angle BAC$ in such way that ω is tangent from the outside to the arc BOC of a circle circumscribed about the triangle BOC, and the circle Ω is tangent internally to a circumcircle of triangle ABC. Prove that the radius of Ω is twice the radius ω.

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1 (original) Given positive real numbers a, b, c, d such that

$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=6$ and $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}=36.$

Prove the inequality

$a^2+b^2+c^2+d^2>ab+ac+ad+bc+bd+cd.$
